Mathematical Modelling and Numerical Analysis Will be set by the publisher
Modélisation Mathématique et Analyse Numérique

A PRIORI ERROR ESTIMATE FOR THE REDUCED HSIEH-CLOUGH-TOCHER DISCRETIZATION OF VISCOSITY IDENTIFICATION IN NAVIER-STOKES EQUATIONS *

Alexandre Vieira [1]

Abstract. We are interested in the problem of identifying the viscosity of a fluid based on observations. This analysis is twofold. First, a stability property of the inverse problem is proved. Secondly, we analyse the discretization of the optimization problem using reduced Hsieh-Clough-Tocher elements, and a convergence with order 3/2 of the identified viscosity with respect to the mesh size is determined. We conclude the paper with some numerical examples showing that this 3/2 order might be enhanced with correct assumptions.

2020 Mathematics Subject Classification. 76M10, 65N21, 65N30, 49M41.

The dates will be set by the publisher.

1. Introduction

We consider the problem of identifying the viscosity in a stationary incompressible Navier-Stokes equations based on observations of the velocity of a fluid. More precisely, we are interested in analyzing the well-posedness of this identification problem, and how it behaves once discretized using reduced Hsieh-Clough-Tocher (rHCT) finite elements. This inverse problem of finding the viscosity based on observation has already gathered interest, mainly due to the possible extension of these results on real-world problems. We cite for instance [15, 21, 23, 24], which are mainly interested in the identification of the viscosity distribution based on observations on the boundary. In our case, we will suppose that we have observations on the whole domain, and we are more interested in studying the effect of the discretization on this parameter identification.

Model studied Let $\Omega \subset \mathbb{R}^2$ be a simply connected open set. We are interested in the steady incompressible Navier-Stokes equations, reading:

$$\begin{aligned} &-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = f \text{ in } \Omega, \\ &\nabla\cdot\mathbf{u} = 0, \\ &\mathbf{u}|_{\partial\Omega} = 0, \end{aligned} \tag{1}$$

Keywords and phrases: Parameter identification, Hsieh-Clough-Tocher, Navier-Stokes equations, error estimate

* *This work has been carried out within the framework of the EUROfusion Consortium, funded by the European Union via the Euratom Research and Training Programme (Grant Agreement No 101052200 — EUROfusion). Views and opinions expressed are however those of the author only and do not necessarily reflect those of the European Union or the European Commission. Neither the European Union nor the European Commission can be held responsible for them.*

[1] Université Côte d'Azur, INRIA, CNRS, LJAD, Nice, France (e-mail: alexandre.vieira@univ-cotedazur.fr)

where $f \in L^2(\Omega)$ is a given source term, $\nabla\cdot$ denotes the divergence, $\mathbf{u}$ is the velocity vector field, p the pressure, and ν the unknown viscosity of the fluid, assumed to be a constant. Throughout this article, we will suppose that $\nu \in [\nu_{\min}, \nu_{\max}]$ for some given $\nu_{\min}$ and $\nu_{\max} > \nu_{\min}$.

In order to take easily into account the incompressibility condition, we use a stream function formulation. We denote $W^{m,p}(\Omega)$ the Sobolev space of functions whose derivatives up to order m is in $L^p(\Omega)$, and we denote $H^k(\Omega) = W^{k,2}(\Omega)$. Using common notations, we denote $H_0^1(\Omega)$ the set of functions in $H^1(\Omega)$ with no-slip boundary condition, meaning the trace on the border of Ω vanishes. Define $\mathcal{V} = \{\psi \in H_0^1(\Omega) \cap H^2(\Omega) \mid \partial_{\mathbf{n}}\psi = 0 \text{ on } \partial\Omega\}$ where $\mathbf{n}$ denotes the outward normal vector to $\partial\Omega$. Using [17, Corollary 3.2], we notice that the operator $\mathbf{curl} = \nabla\times$ defines an isomorphism between $\mathcal{V}$ and $V = \{\mathbf{u} \in H_0^1(\Omega)^2 \mid \nabla \cdot \mathbf{u} = 0\}$. We therefore define the function $\psi \in \mathcal{V}$ such that $\mathbf{u} = \nabla \times \psi$, and define a weak formulation verified by ψ. Following the calculations in [9] (see also [14]), ψ is the solution of the variational equation:

$$\nu \int_\Omega \Delta\psi\Delta\chi - \int_\Omega \Delta\psi[\psi, \chi] = \int_\Omega f\nabla \times \chi, \ \forall\chi \in \mathcal{V}, \tag{2}$$

where $[\psi, \chi] = \partial_x\psi\partial_y\chi - \partial_y\psi\partial_x\chi$ is the Poisson bracket. For scalar functions $\psi, \chi, \phi \in \mathcal{V}$, we will denote $a_0(\psi, \chi) = \int_\Omega \Delta\psi\Delta\chi$ and $a_1(\psi, \phi, \chi) = \int_\Omega -\Delta\psi[\phi, \chi]$. We can show that there exists $\Gamma_1 > 0$ such that

$$a_1(\psi, \chi, \phi) \le \Gamma_1|\psi|_2|\chi|_2|\phi|_2, \ \forall\psi, \chi, \phi \in \mathcal{V}. \tag{3}$$

Due to the antisymmetry of the Poisson bracket, one also has the equality $a_1(\psi, \chi, \chi) = 0$ for all $\psi, \chi \in \mathcal{V}$. Since we will mainly focus on the the evolution of ψ when the viscosity ν changes, we will denote $\nu \mapsto \psi(\nu)$ the operator which assigns ν to the solution ψ of (2).

Discretization We discretize the weak formulation (2) with a finite element method. We will use reduced Hsieh-Clough-Tocher (rHCT) finite elements [10], which are built in order to compute C^1 solutions and were designed for solving fourth order PDEs such as (2). We suppose we are given a family $\{T_h\}_{h>0}$ of shape regular quasi-uniform meshes $T_h = \{K\}$ consisting of closed triangle cells K. The cell parameter h_K is the diameter of K, and we define the mesh parameter h as the maximal cell size, i.e. $h = \max_{K\in T_h} h_K$. Denote $\mathcal{V}_h \subset \mathcal{V}$ the internal approximation of $\mathcal{V}$ using rHCT elements based on the tessellation T_h. We recall an interpolation error result for the interpolant built on the rHCT elements. In the following, we will denote, for $g \in H^k(\Omega)$, the semi norm

$$|g|_k = \left(\sum_{\substack{|\alpha|=k \\ \alpha\in\mathbb{N}}} \|D^\alpha g\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

Theorem 1.1. *[10] Let $v \in H^3(\Omega) \cap \mathcal{V}$. Given a regular family of rHCT triangles, define $\Pi_h v \in \mathcal{V}_h$ the interpolation operator. Then:*

$$|v - \Pi_h v|_m \lesssim h^{3-m}|v|_3, \ m = 0, 1, 2.$$

We now want to solve the approximated equation:

$$\nu_h a_0(\psi_h, \chi_h) + a_1(\psi_h, \psi_h, \chi_h) = \int_\Omega f\nabla \times \chi_h, \ \forall\chi_h \in \mathcal{V}_h, \tag{4}$$

where ν_h is a constant scalar in a neighborhood of ν. Similarly, we will denote $\nu_h \mapsto \psi_h(\nu_h)$ the operator which assigns ν_h to the solution ψ_h of (4). We can show convergence properties of this approximated solution. Theorem 1.1 is used in [14] to prove the convergence of the discretized solution of (4) towards the solution of (2) in the case where $\nu_h = \nu$. This result needs a smoothness assumption on the following helper equation: for $g \in L^2(\Omega)$, let $\zeta \in \mathcal{V}$ be the solution of the equation:

$$\overline{\mathcal{L}}(\zeta, \chi) = \nu a_0(\zeta, \chi) + a_1(\chi, \psi(\nu), \zeta) + a_1(\psi(\nu), \chi, \zeta) = \langle g, \chi\rangle, \ \forall\chi \in \mathcal{V}, \tag{5}$$

where $\langle \cdot, \cdot \rangle$ is the L^2 duality pairing. Equation (5) can be seen as a linearized version of the Navier-Stokes equation.

Throughout this article, we will need the following hypothesis.

(H0): For all $g \in L^2(\Omega)$, one has $\zeta \in H^4(\Omega) \cap \mathcal{V}$ and $|\zeta|_4 \lesssim \|g\|_{L^2(\Omega)}$.

(H1): $f \in L^2(\Omega)$ and $\nu \in [\nu_{\min}, \nu_{\max}]$ where $\nu_{\min} > (\|f\|_{L^2(\Omega)}\Gamma_1)^{\frac{1}{2}}$ and $\nu_{\max} > \nu_{\min}$.

(H2): $\|f\|_{L^2(\Omega)} \neq 0$.

We restate the result concerning the order of convergence for ψ_h here.

Theorem 1.2. *[14] Suppose that **(H0)-(H1)** are verified. Let $\psi = \psi(\nu) \in \mathcal{V}$ be the solution of* (2) *and $\psi_h = \psi_h(\nu) \in \mathcal{V}_h$ be the solution of* (4)*, both associated to the same parameter $\nu \in [\nu_{\min}, \nu_{\max}]$. Assume that $\psi \in H^3(\Omega)$. Then:*

$$\|\psi - \psi_h\|_{L^2(\Omega)} + h^{\frac{1}{2}}\|\nabla(\psi - \psi_h)\|_{L^2(\Omega)} + h\|\Delta(\psi - \psi_h)\|_{L^2(\Omega)} \lesssim h^2.$$

Sketch of proof Based on [9, Theorem 2.2] and using theorem 1.1, one easily proves that $|\psi - \psi_h|_2 \leq ch$ with c independent of ν. In (5), choose $\chi = \psi_h - \psi$ and denote $\zeta_h = \Pi_h \zeta$. After some calculations, one proves that:

$$\begin{aligned} \langle g, \psi_h - \psi \rangle =& a_0(\psi_h - \psi, \zeta - \zeta_h) + a_1(\psi, \psi_h - \psi, \zeta_h - \zeta) \\ &+ a_1(\psi_h - \psi, \psi_h, \zeta - \zeta_h) + a_1(\psi_h - \psi, \psi - \psi_h, \zeta). \end{aligned}$$

Thus, using **(H0)**:

$$\begin{aligned} |\langle g, \psi_h - \psi \rangle| \leq& |\psi_h - \psi|_2 |\zeta - \zeta_h|_2 (\nu_{\max} + \Gamma_1 |\psi|_2 + \Gamma_1 |\psi_h|_2) + \Gamma_1 |\psi_h - \psi|_2^2 |\zeta|_2 \\ \lesssim& h^{2\beta}(|\zeta|_k + |\zeta|_2) \\ \lesssim& h^{2\beta}|g|_0. \end{aligned}$$

Choosing $g = \Delta(\psi - \psi_h)$ and $g = \psi - \psi_h$ yields the desired results. $\square$

Remark 1.1. *As stated in [9], **(H0)** will hold if Ω is a polygon with maximum interior vertex angle lower than 126° ; see [4] for details.*

Viscosity identification Our approach to identify the viscosity parameter ν given some measurement of the velocity on the whole domain will be to minimize a quadratic gap to the observation. More precisely, we will solve the following problem:

$$\begin{aligned} &\min\ J(\nu) = \|\nabla \times \psi(\nu) - \mathbf{u}_{\text{target}}\|^2_{L^2(\Omega)} \\ &\text{s.t.} \begin{cases} \psi(\nu) \text{ solution of (2)}, \\ \nu \in [\nu_{\min}, \nu_{\max}], \end{cases} \end{aligned} \tag{6}$$

for some given $\nu_{\min} > (\Gamma_1 \|f\|_{L^2(\Omega)})^{\frac{1}{2}}$ and $\nu_{\max} > \nu_{\min}$. The solution of the continuous problem will be compared to the solution of its discretized counterpart:

$$\begin{aligned} &\min\ J_h(\nu_h) = \|\nabla \times \psi_h(\nu_h) - \tilde{\Pi}_h \mathbf{u}_{\text{target}}\|^2_{L^2(\Omega)} \\ &\text{s.t.} \begin{cases} \psi_h(\nu_h) \text{ solution of (4)}, \\ \nu_h \in [\nu_{\min}, \nu_{\max}], \end{cases} \end{aligned} \tag{7}$$

where the operator $\tilde{\Pi}_h$ is any interpolation operator for which we will only suppose that $\|\mathbf{u}_{\text{target}} - \tilde{\Pi}_h \mathbf{u}_{\text{target}}\|_{L^2(\Omega)} = O(h^{\frac{3}{2}})$. In the case of reduced HCT elements, $\tilde{\Pi}_h$ can simply be taken as the projection of the reduced HCT interpolator solely on the derivatives (up to a rotation). It should be noted that if, for some $\nu^* \in [\nu_{\min}, \nu_{\max}]$, we have at hand observations $\mathbf{u}_{\text{target}} = \nabla \times \psi(\nu^*)$ (such that $\nabla \mathbf{u}_{\text{target}} \neq 0$) with perfect accuracy (i.e. $\psi(\nu^*)$ is a solution of (2) and no noise is added), then the parameter minimizing (6) can be simply found: in (2), choose $\chi = \psi(\nu^*)$. After simple calculations, one finds

$$\nu^* = \frac{\int_\Omega f \mathbf{u}_{\text{target}}}{\|\nabla \mathbf{u}_{\text{target}}\|^2_{L^2(\Omega)}}.$$

Nonetheless, this supposes that the data $\mathbf{u}_{\text{target}}$ is known perfectly, with no noise, and is a solution of the Navier-Stokes solution for a given viscosity. However, it is rare to have data with no noise or solution of the equations, and we will show that the approach chosen in (6) still works without assumptions on $\mathbf{u}_{\text{target}}$ being solution of a Navier-Stokes equation. Moreover, this result says nothing on the convergence of ν_h, solution of (7), towards ν, solution of (6).

Convergence of the identified parameter Denote ν a solution of (6) and ν_h a solution of (7). The main goal of this paper will be to quantify the error $|\nu - \nu_h|$ for h tending to zero. This study is close to several numerical analysis studies for optimal control problems governed by partial differential equations, see for instance [11–13, 19, 20, 25, 30]. The studies focusing on numerical analysis for parameter identification are more rare. This paper is heavily influenced by two articles. First, Cayco and Nicolaides in [9] analyze the convergence of a pressure recovery algorithm. Their study is focused first on proving an analogue to theorem 1.2 using (non reduced) Hsieh-Clough-Tocher elements, before moving to the pressure reconstruction. Secondly, Rannacher and Vexler in [31] are interested in the identification of scalar parameters in an elliptic linear model using pointwise state observations. In this context, they analyze the discretization of the equation using linear shape functions. They prove the order of convergence of the parameters identified with the discretized problem towards the solution of the continuous problem. Other results in this topic can be found in [5, 22, 28, 29, 33].

Another approach for identifying parameters is to use nudging. The original use of nudging is for system identification, meaning it consists in adding a feedback term in a non-stationary model in order to penalize the deviation from the observed data. However, this needs the model to be known exactly, including all the parameters, except for the initial condition. Azouani, Olson, and Titi in [1] proposed an algorithm in order to adapt the nudging technique to retrieve unknown parameters based on observations. It is now commonly known as the AOT algorithm and has been recently used on the non-stationary Navier-Stokes equations ; see [3, 7, 27].

However, our identification problem (6) lies out of these results for several reasons. First, we use rHCT elements, where most of the literature focus on linear or bilinear C^0 finite elements. Secondly, for the papers focusing on scalar parameters identification, none focus on the stationary Navier-Stokes equation, and the inverse problem is not analyzed as an optimization problem. We also stress the fact that this parameter identification problem is analyzed using the stream function formulation, which must be discretized with an appropriate method since it becomes a fourth order PDE. To the best of our knowledge, a numerical analysis study of a non-linear inverse problem using C^1 conforming elements is still new, exception made of [9] where they use (non reduced) Hsieh-Clough-Tocher elements for pressure identification. The linear case has also gathered only a limited amount of results ; see e.g. [2, 6, 16, 18, 26, 32].

Content The rest of this paper is organized as follows. First, we will analyze the solution map $\nu \mapsto \psi(\nu)$ and prove its derivability and injectivity. This will let us show a Lipschitz property on the inverse problem, which proves a stability property for the problem (6). Secondly, we analyze the convergence of ν_h towards ν and prove its order of convergence. More precisely, we prove, under some hypothesis, that $|\nu - \nu_h| = O(h^{\frac{3}{2}})$ in theorem 3.2. Finally, we conclude this paper with some numerical examples.

In what follows, we will denote $a \lesssim b$ if there exists $C > 0$ (independent of h) such that $a \leq Cb$.

2. Analysis of the solution map

We start our study with the analysis of the solution map $\nu \mapsto \psi(\nu)$. We will mainly be interested in proving its derivability and injectivity. The derivability will be useful in order to analyze (6), since it appears in the derivative of the cost J'. The injectivity will be used to prove the well-posedness of the inverse problem ; more precisely, we will prove that $|\nu - \mu| \lesssim |\psi(\nu) - \psi(\mu)|_1$. Thus, bringing $J(\nu)$ to 0 let us exactly identify the viscosity. Furthermore, it proves that the problem is stable with respect to perturbation of the observation $\mathbf{u}_{\text{target}}$.

2.1. Properties of the solution map

We first recall a result on the boundedness of the solution, exposed in [9, Theorem 2.1].

Theorem 2.1. *Let $f \in L^2(\Omega)$. Denote $\psi = \psi(\nu) \in \mathcal{V}$ the solution of (2) associated to the parameter $\nu \in [\nu_{\min}, \nu_{\max}]$. Then $|\psi|_2 \leq \frac{\|f\|_{L^2(\Omega)}}{\nu}$. Analogously, denote $\psi_h = \psi_h(\nu_h) \in \mathcal{V}_h$ the solution of (4) associated to the parameter $\nu_h \in [\nu_{\min}, \nu_{\max}]$. Then $|\psi_h|_2 \leq \frac{\|f\|_{L^2(\Omega)}}{\nu_h}$.*

This result is useful in order to prove the existence of solution to the linearized Navier-Stokes equations. For the sake of completeness, we prove the existence and uniqueness of solution to (5).

Proposition 2.1. *Suppose* **(H1)** *is verified. For all $g \in L^2(\Omega)$, there exists a unique solution ζ to (5).*

Proof. This is proved using Lax-Milgram theorem. $\overline{\mathcal{L}}$ is obviously bilinear. Using (3), one proves that $\overline{\mathcal{L}}$ is continuous. Concerning the coercivity, note that $a_1(\psi(\nu), \zeta, \zeta) = 0$. Furthermore, using theorem 2.1:

$$\begin{aligned}
\nu a_0(\zeta,\zeta) + a_1(\zeta,\psi(\nu),\zeta) \geq & (\nu - \Gamma_1|\psi(\nu)|_2)|\zeta|_2^2 \\
\geq & \nu\left(1 - \frac{\Gamma_1\|f\|_{L_2(\Omega)}}{\nu^2}\right)|\zeta|_2^2 \\
\geq & \nu_{\max}\left(1 - \frac{\Gamma_1\|f\|_{L_2(\Omega)}}{\nu_{\min}^2}\right)|\zeta|_2^2 \\
\geq & \nu_{\max}|\zeta|_2^2
\end{aligned}$$

Thus, there exists a unique solution ϕ_ν of (5). □

We may now prove the derivability results concerning the solution map.

Theorem 2.2. *Suppose **(H1)** is verified. Denote $\psi(\nu)$ the solution to (2) associated to $\nu \in [\nu_{\min}, \nu_{\max}]$. The application $\psi : \nu \in [\nu_{min}, \nu_{max}] \mapsto \psi(\nu) \in \mathcal{V}$ is continuous and differentiable. Its derivative at point ν is the operator $d\psi_\nu : \mathbb{R} \to \mathcal{V}$ which maps δ to the unique solution ϕ_ν of:*

$$\nu a_0(\phi_\nu, \chi) + a_1(\phi_\nu, \psi(\nu), \chi) + a_1(\psi(\nu), \phi_\nu, \chi) = -\delta a_0(\psi(\nu), \chi), \forall \chi \in \mathcal{V}. \tag{8}$$

Furthermore, the operator $\nu \mapsto d\psi_\nu \in \mathcal{L}(\mathbb{R}, \mathcal{V})$ is continuous.

Proof. Let $\nu \in [\nu_{\min}, \nu_{\max}]$, and let δ such that $\nu + \delta \in [\nu_{\min}, \nu_{\max}]$. We will denote $\psi = \psi(\nu)$ and $\psi_\delta = \psi(\nu + \delta)$. Thus, $\psi - \psi_\delta$ satisfies the equation:

$$\nu \int_\Omega \Delta(\psi - \psi_\delta)\Delta\chi + a_1(\psi - \psi_\delta, \psi, \chi) + a_1(\psi_\delta, \psi - \psi_\delta, \chi) = \delta \int_\Omega \Delta\psi_\delta \Delta\chi, \ \forall \chi \in \mathcal{V}.$$

Let us choose $\chi = \psi - \psi_\delta$. Therefore:

$$\begin{aligned}
\nu|\psi - \psi_\delta|_2^2 &= -a_1(\psi - \psi_\delta, \psi, \psi - \psi_\delta) + \delta \int_\Omega \Delta\psi_\delta \Delta(\psi - \psi_\delta), \\
&\leq \Gamma_1|\psi - \psi_\delta|_2^2|\psi|_2 + \delta|\psi_\delta|_2|\psi - \psi_\delta|_2.
\end{aligned}$$

Using theorem 2.1, and bounding ν^{-1} et $(\nu+\delta)^{-1}$ by $\nu_{\min}^{-1}$, we prove the estimate:

$$\begin{aligned} |\psi-\psi_\delta|_2^2 &\leq \frac{\Gamma_1\|f\|_{L^2(\Omega)}}{\nu_{\min}^2}|\psi-\psi_\delta|_2^2 + \frac{\|f\|_{L^2(\Omega)}}{\nu_{\min}^2}\delta|\psi-\psi_\delta|_2 \\ \iff \left(1-\frac{\Gamma_1\|f\|_{L^2(\Omega)}}{\nu_{\min}^2}\right)|\psi-\psi_\delta|_2 &\leq \frac{\|f\|_{L^2(\Omega)}}{\nu_{\min}^2}\delta. \end{aligned}$$

Since we assumed that $1-\frac{\Gamma_1\|f\|_{L^2(\Omega)}}{\nu_{\min}^2}>0$, it shows that $|\psi-\psi_\delta|_2 = O(\delta)$.

Denote now $\delta\in\mathbb{R}$ fixed and small enough. The proof of the existence of a unique solution to (8) is similar to proposition 2.1 and is therefore omitted. As for theorem 2.1, we can prove that there exists $C>0$ such that $|\phi_\nu|_2 \leq C\delta$. Define $e_\delta = \psi-\psi_\delta-\phi_\nu$, which verifies the equation: for all $\chi\in\mathcal{V}$:

$$\nu\int_\Omega \Delta e_\delta\Delta\chi + a_1(e_\delta,\psi,\chi) + a_1(\psi-\psi_\delta,e_\delta,\chi) = \delta\int_\Omega \Delta(\psi-\psi_\delta)\Delta\chi + a_1(\psi-\psi_\delta,\psi-\psi_\delta,\chi).$$

Testing the equation with $\chi = e_\delta$ yields:

$$\begin{aligned} \nu|e_\delta|_2^2 &= -a_1(e_\delta,\psi,e_\delta) + \delta\int_\Omega \Delta e_\delta\Delta(\psi-\psi_\delta) + a_1(\psi-\psi_\delta,\psi-\psi_\delta,e_\delta), \\ &\leq \Gamma_1|e_\delta|_2^2|\psi|_2 + \delta|\psi-\psi_\delta|_2|e_\delta|_2 + \Gamma_1|\psi-\psi_\delta|_2^2|e_\delta|_2. \end{aligned}$$

Using once again theorem 2.1, one shows that there exists C such that:

$$\left(1-\frac{\Gamma_1\|f\|_{L^2(\Omega)}}{\nu_{\min}^2}\right)|e_\delta|_2 \leq C\delta^2,$$

and thus, $|e_\delta|_2 = O(\delta^2)$.

Eventually, let us show that $\nu\mapsto d\psi_\nu$ is continuous. Take δ fixed and small enough, and $\epsilon\in\mathbb{R}$ such that $\nu+\epsilon$ belongs to $[\nu_{\min},\nu_{\max}]$. Let $\psi_\nu = \psi(\nu)$ (resp. $\psi_\epsilon = \psi(\nu+\epsilon)$) be the solution to (2) associated to ν (resp. $\nu+\epsilon$). Let $\phi_\nu = \phi_\nu(\delta) = d\psi_\nu(\delta)$ (resp. $\phi_\epsilon = \phi_{\nu+\epsilon}(\delta) = d\psi_{\nu+\epsilon}(\delta)$) be the solution of (8) associated to ψ_ν (resp. ψ_ϵ), and define $e_\epsilon = \phi_\nu - \phi_\epsilon$. This function is solution to the equation: for all $\chi\in\mathcal{V}$:

$$\begin{aligned} \nu\int_\Omega \Delta e_\epsilon\Delta\chi &+ a_1(e_\epsilon,\psi_\epsilon,\chi) + a_1(\phi_\nu,\psi_\epsilon-\psi_\nu,\chi) \\ &+ a_1(\psi_\epsilon,e_\epsilon,\chi) + a_1(\psi_\epsilon-\psi_\nu,\phi_\nu,\chi) \\ &= -\delta\int_\Omega \Delta(\psi_\epsilon-\psi_\nu)\Delta\chi - \epsilon\int_\Omega \Delta\psi_\epsilon\Delta\chi. \end{aligned}$$

One chooses $\chi = e_\epsilon$:

$$\begin{aligned} \nu|e_\epsilon|_2^2 =& - a_1(e_\epsilon,\psi_\epsilon,e_\epsilon) - a_1(\phi_\nu,\psi_\epsilon-\psi_\nu,e_\epsilon) - a_1(\psi_\epsilon-\psi_\nu,\phi_\nu,e_\epsilon) \\ &- \delta\int_\Omega \Delta(\psi_\epsilon-\psi_\nu)\Delta e_\epsilon - \epsilon\int_\Omega \Delta\psi_\epsilon\Delta e_\epsilon, \\ \leq& \frac{\Gamma_1\|f\|_{L^2(\Omega)}}{\nu_{\min}}|e_\epsilon|_2^2 + C\delta\epsilon|e_\epsilon|_2 + \epsilon|\psi_\epsilon|_2|e_\epsilon|_2. \end{aligned}$$

With the same calculations as before, this proves that there exists $C>0$ such that:

$$|e_\epsilon|_2 \leq C(\delta\epsilon+\epsilon).$$

Therefore:

$$\|d\psi_{\nu+\epsilon} - d\psi_\nu\|_{\mathcal{L}(\mathbb{R},\mathcal{V})} \leq C \sup_{\delta\leq 1}(\delta\epsilon + \epsilon) \leq 2C\epsilon,$$

thus proving the continuity of the application $\nu \mapsto d\psi_\nu$. □

We now state the injectivity result of the solution map.

Theorem 2.3. *Suppose **(H2)** is verified. Then the operator $\nu \in [\nu_{\min}, \nu_{\max}] \mapsto \psi(\nu) \in H^2(\Omega)$ is injective. Also, for all $\nu \in [\nu_{min}, \nu_{max}]$, the application $\delta \mapsto d\psi_\nu(\delta)$, defined by* (8)*, is injective.*

Proof. Define ν and μ in $[\nu_{\min}, \nu_{\max}]$ such that $\psi(\nu) = \psi(\mu)$ almost everywhere in Ω. Using (2), we have the following equality:

$$(\nu - \mu)\int_\Omega \Delta\psi(\nu)\Delta\chi = 0,\ \forall \chi \in \mathcal{V}.$$

Choosing $\chi = \psi(\nu)$ proves that $(\nu - \mu)|\psi(\nu)|_2^2 = 0$. Suppose that $\nu \neq \mu$. Therefore, $|\psi(\nu)|_2 = 0$, and this implies that $\Delta\psi(\nu) = 0$ almost everywhere in Ω. Going back to the weak formulation (2) verified by $\psi(\nu)$, this implies that $(f, \nabla \times \chi) = 0$ for all $\chi \in \mathcal{V}$. Therefore, one should have $f = 0$ almost everywhere in Ω, which contradicts the hypothesis **(H2)** $\|f\|_{L^2(\Omega)} \neq 0$. Therefore $\mu = \nu$.

The injectivity of $\delta \mapsto d\psi_\nu(\delta)$ is proved similarly using the weak formulation (8). □

2.2. Well-posedness of the inverse problem

Analysis of the optimization problem First of all, we must check that the problem (6) admits a solution. This is proved in the

Proposition 2.2. *Suppose **(H1)** is verified. Then their exists at least one solution to the problem* (6)*.*

Proof. As per theorem 2.2, the application $\nu \in [\nu_{\min}, \nu_{\max}] \mapsto \|\nabla \times \psi(\nu) - \mathbf{u}_{\text{target}}\|^2_{L^2(\Omega)} \in \mathbb{R}$ is continuous (and differentiable), defined on a compact space. Due to Weierstrass' theorem, this application admits a minimum on $[\nu_{\min}, \nu_{\max}]$. □

Analysis of the inverse problem Now, we would like to know if the inverse problem of identifying the viscosity is well defined. This is answered in the two following results.

Proposition 2.3. *Suppose **(H1)** and **(H2)** are verified. Then the following estimate holds:*

$$\forall \nu, \mu \in [\nu_{min}, \nu_{max}],\ |\nu - \mu| \lesssim \|\psi(\nu) - \psi(\mu)\|_{L^2(\Omega)}.$$

Proof. This is an adaptation of the proof of [5, Theorem 2.1]. Let us consider the mapping $\mathcal{T} : (\mu, \delta) \in [\nu_{\min}, \nu_{\max}] \times \mathbb{R} \mapsto d\psi_\mu(\delta) \in \mathcal{V}$ and prove that it is continuous. Choose $\mu, \nu \in [\nu_{\min}, \nu_{\max}]$, $\delta_1, \delta_2 \in \mathbb{R}$.

$$\begin{aligned}|d\psi_\nu(\delta_1) - d\psi_\mu(\delta_2)|_2 \leq& |(d\psi_\nu - d\psi_\mu)(\delta_1)|_2 + |d\psi_\mu(\delta_1 - \delta_2)|_2 \\ \leq& \|d\psi_\nu - d\psi_\mu\|_{\mathcal{L}(\mathbb{R};\mathcal{V})}|\delta_1| + \|d\psi_\nu\|_{\mathcal{L}(\mathbb{R};\mathcal{V})}|\delta_1 - \delta_2|.\end{aligned}$$

Using theorem 2.2, one proves the continuity of $\mathcal{T}$.

By the injectivity of $\delta \mapsto d\psi_\nu(\delta)$ proved in theorem 2.3, there exists $c > 0$ such that:

$$\|d\psi_\nu(\delta)\|_{L^2(\Omega)} \geq c|\delta|,\ \forall \delta \in \mathbb{R},\ \forall \mu \in [\nu_{\min}, \nu_{\max}].$$

Due to the continuity of $\mathcal{T}$, for all $\varepsilon > 0$, if $\nu, \mu \in [\nu_{\min}, \nu_{\max}]$ satisfy $|\nu - \mu| \leq \varepsilon$, then there exists some $c' > 0$ such that:

$$\|(d\psi_\nu - d\psi_\mu)(\delta)\|_{L^2(\Omega)} \lesssim |(d\psi_\nu - d\psi_\mu)(\delta)|_2 \leq c'\varepsilon|\delta|.$$

We choose $\varepsilon = \frac{c}{2c'}$ in order to get $\|(d\psi_\nu - d\psi_\mu)(\delta)\|_{L^2(\Omega)} \leq \frac{c}{2}|\delta|$. Let us take $\mu, \nu \in [\nu_{\min}, \nu_{\max}]$. We differentiate two cases:

- Suppose that $|\mu-\nu|<\varepsilon$. Denote $\delta=\mu-\nu$. We have that:

$$\psi(\mu)-\psi(\nu)=\int_0^1 \frac{d}{ds}\psi(\nu+s\delta)ds=d\psi_\nu(\delta)+\int_0^1(d\psi_{\nu+s\delta}-d\psi_\nu)(\delta)ds,$$

and therefore:

$$\|\psi(\mu)-\psi(\nu)\|_{L^2(\Omega)}\geq \frac{c}{2}|\mu-\nu|.$$

- Consider now the case $|\mu-\nu|\geq\varepsilon$. Due to the injectivity of ψ proved in theorem 2.3, the minimum m of the continuous map $(\mu,\nu)\mapsto\|\psi(\mu)-\psi(\nu)\|_{L^2(\Omega)}$ on the compact set $\mathcal{U}=\{(\mu,\nu)\in[\nu_{\min},\nu_{\max}]^2\ ;\ |\nu-\mu|\geq\varepsilon\}$ is positive: $m>0$. Furthermore, for all $\mu,\nu\in\mathcal{U}$, $\|\psi(\mu)-\psi(\nu)\|_{L^2(\Omega)}\geq m\geq\frac{m}{d}|\mu-\nu|$, where d is the diameter of $\mathcal{U}$.

If one takes $C=\max\left(\frac{2}{c},\frac{d}{m}\right)$, this proves that, for all $\mu,\nu\in[\nu_{\min},\nu_{\max}]$, $|\mu-\nu|\leq C\|\psi(\mu)-\psi(\nu)\|_{L^2(\Omega)}$. □

Corollary 2.1. *Suppose **(H1)** and **(H2)** are verified. Then the following estimate holds:*

$$\forall\nu,\mu\in[\nu_{min},\nu_{max}],\ |\nu-\mu|\lesssim\|\nabla\times\psi(\nu)-\nabla\times\psi(\mu)\|_{L^2(\Omega)}.$$

Proof. Simply remark that $\|\nabla\times\psi(\nu)-\nabla\times\psi(\mu)\|_{L^2(\Omega)}=\|\nabla\psi(\nu)-\nabla\psi(\mu)\|_{L^2(\Omega)}$ and use Poincaré's inequality on proposition 2.3. □

We underline the fact that all the results of theorem 2.2, theorem 2.3 and corollary 2.1 can be simply adapted to the discrete case (7).

3. Stability with the discretization

This section is devoted to the analysis of the solutions of (6) and (7) with respect to the discretization process. More precisely, we will prove that $|\nu-\nu_h|=O(h^{\frac{3}{2}})$. In order to prove this result, we will need to prove an analogue to the theorem 1.2 for the derivative maps $\nu\mapsto\psi'(\nu)$ and $\nu\mapsto\psi''(\nu)$.

3.1. A stability theorem

The analysis of the discretization will focus on the application of the following proposition 3.1. It is adapted from [31, Theorem 3.1] to the case of box constraints, as it is the case in (6). Note that the second order optimality conditions for (6) need the notion of critical cone at $\bar\nu\in[\nu_{\min},\nu_{\max}]$, which is expressed as:

$$C_{\bar\nu}^{J'}=\{\delta\in\mathbb{R}|\delta=\lambda(\nu-\bar\nu)\text{ such that }\nu\in[\nu_{\min},\nu_{\max}],\ \lambda>0,\ J'(\bar\nu)\delta=0\}.$$

Note that if $\bar\nu\in(\nu_{\min},\nu_{\max})$ and $J'(\bar\nu)=0$, then $C_{\bar\nu}^{J'}=\mathbb{R}$. We recall the first and second order necessary conditions of optimality.

Theorem 3.1. *[8, Theorem 3.7 & 3.8] Let $\mathcal{J}:[\nu_{\min},\nu_{\max}]\to\mathbb{R}$ be a twice differentiable function and consider the optimization problem $\min_{\nu\in[\nu_{\min},\nu_{\max}]}\mathcal{J}(\nu)$. Denote ν^* a local solution, meaning $\mathcal{J}(\nu^*)\leq\mathcal{J}(\nu)$ for all $\nu\in[\nu_{\min},\nu_{\max}]$ in a neighborhood of ν^*. Then*

$$\mathcal{J}'(\nu^*)(\nu-\nu^*)\geq 0,\ \forall\nu\in[\nu_{\min},\nu_{\max}],\tag{9}$$

and

$$\mathcal{J}''(\nu^*)\delta^2\geq 0,\ \forall\delta\in C_{\nu^*}^{\mathcal{J}'}.\tag{10}$$

If $\nu^*\in[\nu_{\min},\nu_{\max}]$ satisfies the variational inequality (9) *and the second order sufficient condition*

$$\mathcal{J}''(\nu^*)\delta^2>0,\ \forall\delta\in C_{\nu^*}^{\mathcal{J}'}\backslash\{0\},\tag{11}$$

then ν^* is a strict local optimal solution, meaning $\mathcal{J}(\nu^*) < \mathcal{J}(\nu)$ for all $\nu \in [\nu_{\min}, \nu_{\max}]$ in a neighborhood of ν^*.

As shown in [20, Lemma 1.12], (9) is equivalent to the existence of multipliers μ_1, μ_2 such that:

$$\begin{aligned}
&\mathcal{J}'(\nu) - \mu_1 + \mu_2 = 0,\\
&0 \le \mu_1 \perp \nu - \nu_{\min} \ge 0,\\
&0 \le \mu_2 \perp \nu_{\max} - \nu \ge 0,
\end{aligned}$$

where $0 \le a \perp b \ge 0$ means that $a \ge 0$, $b \ge 0$ and $ab = 0$. We may now prove the proposition 3.1. This is a variant of the perturbation theorems for differentiable mappings, but with the addition of complementarity constraints. Note that $F = J'$ satisfy (12) at the optimal solution ν of (6), since it is exactly the first order condition of optimality of (6).

Proposition 3.1. *Let $F, F_h : [\nu_{\min}, \nu_{\max}] \to \mathbb{R}$, for a given parameter $h > 0$, be continuous and differentiable functions. Suppose there is $\nu \in [\nu_{\min}, \nu_{\max}]$ and multipliers $\mu_1, \mu_2 \in \mathbb{R}$ such that:*

$$\begin{aligned}
&F(\nu) - \mu_1 + \mu_2 = 0,\\
&0 \le \mu_1 \perp \nu - \nu_{\min} \ge 0,\\
&0 \le \mu_2 \perp \nu_{\max} - \nu \ge 0.
\end{aligned} \tag{12}$$

Suppose the following holds:

1. *The derivative $F'(\nu)$ is positive on the critical cone, i.e. there exists $\gamma > 0$ such that*
$$F'(\nu)\delta^2 \ge \gamma\delta^2,\ \forall \delta \in C_\nu^F. \tag{13}$$
2. *There exists a neighborhood of ν (denoted $\mathcal{U}(\nu)$) and a positive number $L(h)$ such that, for all $\nu_1, \nu_2 \in \mathcal{U}(\nu)$:*
$$|F_h'(\nu_1) - F_h'(\nu_2)| \le L(h)|\nu_1 - \nu_2|. \tag{14}$$
3. *The following limit on F_h holds*
$$\lim_{h \to 0} L(h)|F(\nu) - F_h(\nu)| = 0. \tag{15}$$
4. *The following limit on F_h' holds*
$$\lim_{h \to 0} |F'(\nu) - F_h'(\nu)| = 0. \tag{16}$$

Then, given h small enough, there exist $\nu_h \in [\nu_{\min}, \nu_{\max}]$ in a neighborhood of ν and multipliers $\mu_1^h, \mu_2^h \in \mathbb{R}$ such that:

$$\begin{aligned}
&F_h(\nu_h) - \mu_1^h + \mu_2^h = 0,\\
&0 \le \mu_1^h \perp \nu_h - \nu_{\min} \ge 0,\\
&0 \le \mu_2^h \perp \nu_{\max} - \nu_h \ge 0.
\end{aligned} \tag{17}$$

Furthermore, $F_h'(\nu_h)$ is positive on the critical cone uniformly in h, and we have the a priori error estimate:

$$|\nu - \nu_h| \le \frac{2}{\gamma}|F(\nu) - F_h(\nu)|. \tag{18}$$

Proof. Remark first that if $\nu \in (\nu_{\min}, \nu_{\max})$, then (12) boils down to $F(\nu) = 0$, $\mu_1 = \mu_2 = 0$, and $C_\nu^F = \mathbb{R}$. For some $\rho = \frac{\gamma}{kL(h)}$ with $k \geq 4$ sufficiently large, one has:

$$B_\rho(\nu) = \{\tau \in [\nu_{\min}, \nu_{\max}],\ |\nu - \tau| \leq \rho\} \subset \mathcal{U}(\nu) \cap (\nu_{\min}, \nu_{\max}).$$

Since the whole proof of [31, Theorem 3.1] is based on this neighborhood $B_\rho(\nu)$, the proof in the case $\nu \in (\nu_{\min}, \nu_{\max})$ is identical.

Suppose that $\nu = \nu_{\min}$ (the proof is the same in the case $\nu = \nu_{\max}$). In this case, remark that necessarily, $\mu_2 = 0$ and $F(\nu_{\min}) = \mu_1 \geq 0$. Let $\tau \in B_\rho(\nu)$. With similar arguments as the one in [31, Theorem 3.1], we show that for all $\delta \in C_\tau^F$, $F_h'(\tau)\delta^2 \geq \frac{\gamma}{2}\delta^2$. Let us prove that there exists $\nu_h \in B_\rho(\nu)$ verifying (17). We distinguish two cases:

- Either $F(\nu_{\min}) > 0$, and in this case, due to (15), for h small enough, $F_h(\nu_{\min}) > 0$. Choose then $\mu_1^h = F_h(\nu_{\min})$, and conclude the proof by choosing $\nu_h = \nu_{\min}$.
- Or $F(\nu_{\min}) = 0$. In this case, we distinguish once more two different cases:
  - Either $F_h(\nu_{\min}) \geq 0$, and we choose $\mu_1 = F_h(\nu_{\min})$, and this concludes the proof by choosing $\nu_h = \nu_{\min}$.
  - Or $F_h(\nu_{\min}) < 0$. In this case, let $\bar{\nu} \in B_\rho(\nu_{\min})$. Due to the mean value theorem, remark that there exists $\tau \in B_\rho(\nu)$ such that

    $$F_h(\nu) - F_h(\nu_{\min}) = F_h'(\tau)(\nu - \nu_{\min}) \geq \frac{\gamma}{2}(\nu - \nu_{\min}),$$

    where we remark that $\nu - \nu_{\min} \in C_\tau^F$. Due to (15), for a given $\varepsilon > 0$ and for all h, both small enough, one has $F_h(\nu_{\min}) \geq -\frac{\gamma}{2}(\nu - \nu_{\min}) + \varepsilon$. Therefore $F_h(\nu) \geq \frac{\gamma}{2}(\nu - \nu_{\min}) + F_h(\nu_{\min}) \geq \varepsilon > 0$. Since F_h is continuous, using the intermediate value theorem, there exists $\nu_h \in B_\rho(\nu_{\min})$ such that $F_h(\nu_h) = 0$. Since F_h' is positive on this interval, it implies the uniqueness of ν_h in $B_\rho(\nu_{\min})$.

The proof of the estimate (18) is done similarly as in [31, Theorem 3.1]. □

3.2. **Derivative computation and properties**

Proposition 3.1 will be used with the functions $F = J'$ and $F_h = J_h'$. We therefore need to compute the derivatives of J.

Proposition 3.2. *Suppose* **(H1)** *holds. One proves that:*

$$J'(\nu) = \int_\Omega \nabla \times \psi'(\nu) \cdot (\nabla \times \psi(\nu) - \mathbf{u}_{target}),$$

where $\psi'(\nu) = d\psi_\nu(1)$ *is defined in* (8)*, and*

$$J''(\nu) = \int_\Omega \nabla \times \psi''(\nu)(\nabla \times \psi(\nu) - \mathbf{u}_{target}) + \|\nabla \times \psi'(\nu)\|^2_{L^2(\Omega)},$$

where $\psi''(\nu)$ *is the unique solution of the following equation: for all* $\chi \in \mathcal{V}$*:*

$$\begin{aligned} \nu a_0(\psi''(\nu), \chi) + a_1(\psi''(\nu), \psi(\nu), \chi) + a_1(\psi(\nu), \psi''(\nu), \chi) \\ = -2(a_0(\psi'(\nu), \chi) + a_1(\psi'(\nu), \psi'(\nu), \chi)). \end{aligned} \tag{19}$$

Proof. The proof of the existence and uniqueness of solution to (19) is similar to proposition 2.1 and is thus omitted. The first and second derivatives of J simply come from the chain rule. We only give the details for the computation of the second derivative $\psi''(\nu)$, since the calculations for the first derivative are similar and already done in theorem 2.2. In the following, χ will simply be any element of $\mathcal{V}$.

$$\begin{aligned}
0 = \lim_{\delta\to 0} \delta^{-1}(&a_0(\psi(\nu+\delta)-\psi(\nu),\chi) + \nu a_0(\psi'(\nu+\delta)-\psi'(\nu),\chi) + \delta a_0(\psi'(\nu+\delta),\chi) \\
&+ a_1(\psi(\nu+\delta),\psi'(\nu+\delta),\chi) + a_1(\psi'(\nu+\delta),\psi(\nu+\delta),\chi)) \\
&- a_1(\psi(\nu),\psi'(\nu),\chi) - a_1(\psi'(\nu),\psi(\nu),\chi))) \\
= a_0(\psi'(\nu),\chi) + \lim_{\delta\to 0}\delta^{-1}(&a_0(\psi(\nu+\delta)-\psi(\nu),\chi) + \nu a_0(\psi'(\nu+\delta)-\psi'(\nu),\chi) \\
&+ a_1(\psi(\nu+\delta),\psi'(\nu+\delta),\chi) - a_1(\psi(\nu+\delta),\psi'(\nu),\chi) \\
&+ a_1(\psi(\nu+\delta),\psi'(\nu),\chi) - a_1(\psi(\nu),\psi'(\nu),\chi) \\
&+ a_1(\psi'(\nu+\delta),\psi(\nu+\delta),\chi) - a_1(\psi'(\nu),\psi(\nu+\delta),\chi) \\
&+ a_1(\psi'(\nu),\psi(\nu+\delta),\chi) - a_1(\psi'(\nu),\psi(\nu),\chi))) \\
= 2a_0(\psi'(\nu),\chi) + &\nu a_0(\psi''(\nu),\chi) + a_1(\psi(\nu),\psi''(\nu),\chi) \\
&+ 2a_1(\psi'(\nu),\psi'(\nu),\chi) + a_1(\psi''(\nu),\psi(\nu),\chi).
\end{aligned}$$

□

Similarly, one proves that:

$$J_h'(\nu) = \int_\Omega \nabla\times\psi_h'(\nu)\cdot(\nabla\times\psi_h(\nu) - \tilde{\Pi}_h\mathbf{u}_{\text{target}}),$$

$$J_h''(\nu) = \int \nabla\times\psi_h''(\nu)(\nabla\times\psi_h(\nu) - \tilde{\Pi}_h\mathbf{u}_{\text{target}}) + \|\nabla\times\psi'(\nu)\|^2_{L^2(\Omega)},$$

where $\psi_h'(\nu)$ and $\psi_h''(\nu)$ are defined as the solutions of the variational equations:

$$\begin{aligned}
\nu a_0(\psi_h'(\nu),\chi) + a_1(\psi_h'(\nu),\psi_h(\nu),\chi) + a_1(\psi_h(\nu),\psi_h'(\nu),\chi) &= -a_0(\psi_h(\nu),\chi), \\
\nu a_0(\psi_h''(\nu),\chi) + a_1(\psi_h''(\nu),\psi_h(\nu),\chi) + a_1(\psi_h(\nu),\psi_h''(\nu),\chi)& \\
&= -2(a_0(\psi_h'(\nu),\chi) + a_1(\psi_h'(\nu),\psi_h'(\nu),\chi)),
\end{aligned}$$

for all $\chi\in\mathcal{V}_h$.

Properties of the derivatives Given the estimate of theorem 1.2, we would like to prove the same kind of estimates for $\psi'(\nu)$ et $\psi''(\nu)$. This will be used in order to check that $F = J'$ and $F_h = J_h'$ comply the assumptions of proposition 3.1.

Proposition 3.3. *Suppose **(H0)-(H1)** hold. Let $\nu\in[\nu_{\min},\nu_{\max}]$ be fixed. Denote $\psi(\nu)\in\mathcal{V}$ the solution of (2). Suppose also that $\psi(\nu)$, $\psi'(\nu)$ and $\psi''(\nu)$ are in $H^3(\Omega)$. Then*

$$|\psi'(\nu) - \psi_h'(\nu)|_2 = O(h),$$

$$|\psi''(\nu) - \psi_h''(\nu)|_2 = O(h).$$

Proof. Since ψ_h (resp. ψ_h') appears in the weak formulation satisfied by ψ_h' (resp. ψ_h''), the error needs to be split in two in order to see the influence of the discretization of the equation, and the influence of the discretization ψ_h of ψ (resp. ψ_h' of ψ'). Let us start with $\psi'(\nu)$. Define $\bar{\psi}_h'(\nu)\in\mathcal{V}_h$ as the solution of the equation: $\forall\chi_h\in\mathcal{V}_h$

$$\nu a_0(\bar{\psi}_h'(\nu),\chi_h) + a_1(\bar{\psi}_h',\psi(\nu),\chi_h) + a_1(\psi(\nu),\bar{\psi}_h',\chi_h) = -a_0(\psi(\nu),\chi_h).$$

Let us split the error: $\psi'(\nu) - \psi_h'(\nu) = \psi'(\nu) - \bar{\psi}_h'(\nu) + \bar{\psi}_h'(\nu) - \psi_h'(\nu)$.

- We first focus on $\delta\psi' = \bar{\psi}'_h(\nu) - \psi'_h(\nu)$. We will denote $\delta\psi = \psi(\nu) - \psi_h(\nu)$. We have the following equality: $\forall \chi_h \in \mathcal{V}_h$

$$\begin{aligned}\nu a_0(\delta\psi', \chi_h) &+ a_1(\delta\psi', \psi(\nu), \chi_h) + a_1(\psi(\nu), \delta\psi', \chi_h) \\ &= -a_0(\delta\psi, \chi_h) - a_1(\psi'_h(\nu), \delta\psi, \chi_h) - a_1(\delta\psi, \psi'_h(\nu), \chi_h).\end{aligned}$$

We choose $\chi_h = \delta\psi' \in \mathcal{V}_h$, which gives us the estimate

$$(\nu - \Gamma_1|\psi(\nu)|_2)|\delta\psi'|_2^2 \leq (2\Gamma_1|\psi'_h(\nu)|_2 + 1)|\delta\psi|_2|\delta\psi'|_2.$$

Therefore, using the fact that $|\psi(\nu)|_2$ and $|\psi'_h(\nu)|_2$ are bounded, one proves using theorem 1.2 that:

$$|\delta\psi'|_2 \lesssim |\delta\psi|_2 \lesssim h.$$

- Let us now focus on $\psi'(\nu) - \bar{\psi}'_h(\nu)$. The following relation holds for all $\chi_h \in \mathcal{V}_h$:

$$\begin{aligned}\mathcal{L}(\psi'(\nu) - \bar{\psi}'_h(\nu), \chi_h) =& \nu a_0(\psi'(\nu) - \bar{\psi}'_h(\nu), \chi_h) + a_1(\psi'(\nu) - \bar{\psi}'_h(\nu), \psi(\nu), \chi_h) \\ &+ a_1(\psi(\nu), \psi'(\nu) - \bar{\psi}'_h(\nu), \chi_h) \\ =& 0.\end{aligned}$$

Therefore, for all $\chi_h \in \mathcal{V}_h$:

$$\begin{aligned}\mathcal{L}(\psi'(\nu) - \bar{\psi}'_h(\nu), \psi'(\nu) - \bar{\psi}'_h(\nu)) =& \mathcal{L}(\psi'(\nu) - \bar{\psi}'_h(\nu), \psi'(\nu)) \\ &- \mathcal{L}(\psi'(\nu) - \bar{\psi}'_h(\nu), \bar{\psi}'_h(\nu)) \\ =& \mathcal{L}(\psi'(\nu) - \bar{\psi}'_h(\nu), \psi'(\nu)) \\ &- \mathcal{L}(\psi'(\nu) - \bar{\psi}'_h(\nu), \chi_h) \\ =& \mathcal{L}(\psi'(\nu) - \bar{\psi}'_h(\nu), \psi'(\nu) - \chi_h).\end{aligned}$$

It implies:

$$(\nu - \Gamma_1|\psi(\nu)|_2)|\psi'(\nu) - \bar{\psi}'_h(\nu)|_2^2 \leq (\nu + 2\Gamma_1|\psi(\nu)|_2)|\psi'(\nu) - \bar{\psi}'_h(\nu)|_2|\psi'(\nu) - \chi_h|_2.$$

Using the same bounds on ν and $|\psi|_2$ as before, and using theorem 1.1, one proves that:

$$|\psi'(\nu) - \bar{\psi}'_h(\nu)|_2 \lesssim \inf_{\chi_h \in \mathcal{V}_h} |\psi'(\nu) - \chi_h|_2 \lesssim h|\psi'(\nu)|_3.$$

The proof concerning $\psi''(\nu) - \psi''_h(\nu)$ is similar and found in the Appendix. □

Proposition 3.4. *Suppose **(H0)**-**(H1)** are verified. Let $\nu \in [\nu_{\min}, \nu_{\max}]$ be fixed. Suppose also that*

- *$\psi(\nu) \in H^3(\Omega) \cap \mathcal{V}$ and $\psi_h(\nu) \in H^3(\Omega) \cap \mathcal{V}_h$.*
- *$\psi'(\nu), \psi''(\nu) \in H^4(\Omega) \cap \mathcal{V}$ and $\psi'_h(\nu), \psi''_h(\nu) \in H^4(\Omega) \cap \mathcal{V}_h$.*

Then

$$\begin{aligned}h^{\frac{1}{2}}|\psi'(\nu) - \psi'_h(\nu)|_1 + \|\psi'(\nu) - \psi'_h(\nu)\|_{L^2(\Omega)} = O(h^2), \\ h^{\frac{1}{2}}|\psi''(\nu) - \psi''_h(\nu)|_1 + \|\psi''(\nu) - \psi''_h(\nu)\|_{L^2(\Omega)} = O(h^2).\end{aligned}$$

Proof. We remind the reader that ζ is the solution of the equation (5). Denote $\zeta_h = \Pi_h\zeta$ and $\delta\psi = \psi(\nu) - \psi_h(\nu)$. In (5), take $\chi = \psi'(\nu) - \psi'_h(\nu)$:

$$\langle g, \psi'(\nu) - \psi'_h(\nu)\rangle = \overline{\mathcal{L}}(\zeta - \zeta_h, \psi'(\nu) - \psi'_h(\nu)) + \overline{\mathcal{L}}(\zeta_h, \psi'(\nu) - \psi'_h(\nu)).$$

Note that
$$\overline{\mathcal{L}}(\zeta_h, \psi'(\nu) - \psi_h'(\nu)) = -(a_0(\delta\psi, \zeta_h) + a_1(\psi_h'(\nu), \delta\psi, \zeta_h) + a_1(\delta\psi, \psi_h'(\nu), \zeta_h)).$$
Therefore,
$$\begin{aligned}\langle g, \psi'(\nu) - \psi_h'(\nu)\rangle =& \overline{\mathcal{L}}(\zeta - \zeta_h, \psi'(\nu) - \psi_h'(\nu)) + a_0(\delta\psi, \zeta - \zeta_h)\\ &+ a_1(\psi_h'(\nu), \delta\psi, \zeta - \zeta_h) + a_1(\delta\psi, \psi_h'(\nu), \zeta - \zeta_h)\\ &- (a_0(\delta\psi, \zeta) + a_1(\psi_h'(\nu), \delta\psi, \zeta) + a_1(\delta\psi, \psi_h'(\nu), \zeta)).\end{aligned}$$
Using integration by parts and Sobolev inclusions, one shows that:
$$a_0(\delta\psi, \zeta) = \int_\Omega \delta\psi \Delta^2 \zeta \lesssim \|\delta\psi\|_{L^2(\Omega)} |\zeta|_4,$$
$$\begin{aligned}a_1(\psi_h'(\nu), \delta\psi, \zeta) =& -\int_\Omega \Delta\psi_h'(\nu) \nabla \times \zeta \cdot \nabla\delta\psi\\ =& \int_\Omega \nabla \cdot (\Delta\psi_h'(\nu)\nabla \times \zeta)\delta\psi\\ \leq& \|\delta\psi\|_{L^2(\Omega)} \|\zeta\|_{W^{2,4}(\Omega)} \|\psi_h'(\nu)\|_{W^{3,4}(\Omega)}\\ \lesssim& \|\delta\psi\|_{L^2(\Omega)} |\zeta|_3 |\psi_h'(\nu)|_4,\\ a_1(\delta\psi, \psi_h'(\nu), \zeta) =& \int_\Omega \delta\psi \Delta([\psi_h'(\nu), \zeta])\\ \leq& \|\delta\psi\|_{L^2(\Omega)} \|\zeta\|_{W^{3,4}(\Omega)} \|\psi_h'(\nu)\|_{W^{3,4}(\Omega)}\\ \lesssim& \|\delta\psi\|_{L^2(\Omega)} |\zeta|_4 |\psi_h'(\nu)|_4.\end{aligned}$$
Using the hypothesis $|\zeta|_4 \lesssim \|g\|_{L^2(\Omega)}$, we get:
$$\begin{aligned}\langle g, \psi'(\nu) - \psi_h'(\nu)\rangle \lesssim& (\nu + 2\Gamma_1 |\psi(\nu)|_2)|\zeta - \zeta_h|_2 |\psi'(\nu) - \psi_h'(\nu)|_2\\ &+ (1 + 2\Gamma_1 |\psi_h'(\nu)|_2)|\zeta - \zeta_h|_2 |\psi(\nu) - \psi_h(\nu)|_2\\ &+ (1 + 2|\psi_h'(\nu)|_4)\|\delta\psi\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.\end{aligned}$$

Using the interpolation error between ζ and ζ_h (see theorem 1.1) and the estimate on $|\psi(\nu) - \psi_h(\nu)|_2$ and $|\psi'(\nu) - \psi_h'(\nu)|_2$ (see theorem 1.2 and proposition 3.3), we have:
$$\begin{aligned}\langle g, \psi'(\nu) - \psi_h'(\nu)\rangle \lesssim& h^2|\zeta|_4 + h^2|\zeta|_4 + h^2\|g\|_{L^2(\Omega)}\\ \lesssim& h^2\|g\|_{L^2(\Omega)}.\end{aligned}$$

We choose now $g = \Delta(\psi'(\nu) - \psi_h'(\nu))$, we get:
$$|\psi'(\nu) - \psi_h'(\nu)|_1^2 = \langle g, \psi'(\nu) - \psi_h'(\nu)\rangle \lesssim h^2|\psi'(\nu) - \psi_h'(\nu)|_2 \lesssim h^3.$$
This implies:
$$|\psi'(\nu) - \psi_h'(\nu)|_1 \lesssim h^{\frac{3}{2}}.$$
Choosing now $g = \psi'(\nu) - \psi_h'(\nu)$, we prove straight away that:
$$\|\psi'(\nu) - \psi_h'(\nu)\|_{L^2(\Omega)} \lesssim h^2.$$

The proof for $\psi''(\nu) - \psi_h''(\nu)$ is similar and done in the Appendix. □

Remark 3.1. *As for remark 1.1 concerning **(H0)**, the assumptions ψ' and ψ'' in $H^4(\Omega)$ will hold if Ω is a polygon with maximum interior vertex angle lower than 126° and $g = -\Delta^2\psi(\nu) \in L^2(\Omega)$.*

We may now prove the order of convergence for the parameter identification problem. Note that we suppose that the solution is *stable* (in the terminology of [31]), meaning that the sufficient condition of optimality (13) is verified with $F = J'$. We first need the following lemma in order to prove the lipschitz condition (14).

Lemma 3.1. *Assume **(H0)**. Let $\nu \in [\nu_{\min}, \nu_{\max}]$ and let $\delta \in \mathbb{R}$ be such that $\nu + \delta \in [\nu_{\min}, \nu_{\max}]$. Then one has the estimate $|\psi''(\nu) - \psi''(\nu+\delta)|_2 \lesssim |\delta|$.*

Proof. First, note that one can easily prove that, for all $\mu \in [\nu_{\min}, \nu_{\max}]$, $|\psi''(\mu)|_2 \leq C$ for some $C > 0$. The proof simply consists in choosing $\chi = \psi''(\mu)$ in (19).

We will denote $\psi = \psi(\nu)$, $\psi_\delta = \psi(\nu + \delta)$ (the same definition holds for ψ' and ψ''). The function $\psi'' - \psi''_\delta$ solves the equation: for all $\chi \in \mathcal{V}$:

$$\begin{aligned}\nu a_0(\psi'' - \psi''_\delta, \chi) + a_1(\psi'' - \psi''_\delta, \psi, \chi) + a_1(\psi, \psi'' - \psi''_\delta, \chi)&\\ =&\delta a_0(\psi''_\delta, \chi) - a_1(\psi''_\delta, \psi - \psi_\delta, \chi) - a_1(\psi - \psi_\delta, \psi''_\delta, \chi)\\ &- 2(a_0(\psi' - \psi'_\delta, \chi) + a_1(\psi' - \psi'_\delta, \psi', \chi) + a_1(\psi'_\delta, \psi' - \psi'_\delta, \chi)).\end{aligned}$$

We choose $\chi = \psi'' - \psi''_\delta$:

$$\begin{aligned}(\nu_{\min} - \Gamma_1 \nu_{\min}^{-1} \|f\|_{L^2(\Omega)})|\psi'' - \psi''_\delta|_2^2 \leq &[\delta|\psi''_\delta|_2 + 2\Gamma_1|\psi''_\delta|_2|\psi - \psi_\delta|_2\\ &+ 2(1 + \Gamma_1(|\psi'|_2 + |\psi'_\delta|_2))\\ &|\psi' - \psi'_\delta|_2]|\psi'' - \psi''_\delta|_2.\end{aligned}$$

Using theorem 2.2, this implies that $|\psi'' - \psi''_\delta|_2 \lesssim |\delta|$. □

Theorem 3.2. *Suppose **(H0)-(H2)** are verified, that the solution ν of (6) is stable, and that*

- *$\psi(\nu) \in H^3(\Omega) \cap \mathcal{V}$ and $\psi_h(\nu) \in H^3(\Omega) \cap \mathcal{V}_h$.*
- *$\psi'(\nu), \psi''(\nu) \in H^4(\Omega) \cap \mathcal{V}$ and $\psi'_h(\nu), \psi''_h(\nu) \in H^4(\Omega) \cap \mathcal{V}_h$.*

Suppose also that $\|\mathbf{u}_{target} - \tilde{\Pi}_h \mathbf{u}_{target}\|_{L^2(\Omega)} = O(h^{\frac{3}{2}})$. Denote ν_h the solution of (7). Then one has the estimate $|\nu - \nu_h| = O(h^{\frac{3}{2}})$.

Proof. For some $\mu \in [\nu_{\min}, \nu_{\max}]$, we will denote $\delta\psi(\mu) = \psi(\mu) - \psi_h(\mu)$, $\delta\psi'(\mu) = \psi'(\mu) - \psi'_h(\mu)$, $\delta\psi''(\mu) = \psi''(\mu) - \psi''_h(\mu)$. The proof consists only in the application of proposition 3.1, where we use $F = J'$, $F_h = J'_h$. Condition (14) is proved using lemma 3.1. Note that, as proved in proposition 3.2 and in proposition 3.4:

$$\begin{aligned}|J'(\mu) - J'_h(\mu)| =&|\langle \nabla \times \delta\psi'(\mu), \nabla \times \psi(\mu) - \mathbf{u}_{\text{target}}\rangle + \langle \nabla \times \psi'_h(\mu), \nabla \times \delta\psi(\mu)\rangle\\ &- \langle \nabla \times \psi'_h(\mu), \mathbf{u}_{\text{target}} - \tilde{\Pi}_h \mathbf{u}_{\text{target}}\rangle|\\ \leq&|\delta\psi'(\mu)|_1 \|\nabla \times \psi(\mu) - \mathbf{u}_{\text{target}}\|_{L^2(\Omega)} + |\psi'_h(\mu)|_1 |\delta\psi(\mu)|_1\\ &+ |\psi'_h(\mu)|_1 \|\mathbf{u}_{\text{target}} - \tilde{\Pi}_h \mathbf{u}_{\text{target}}\|_{L^2(\Omega)}\\ \lesssim&h^{\frac{3}{2}}.\end{aligned}$$

$$\begin{aligned}|J''(\mu) - J''_h(\mu)| =&|\langle \nabla \times \delta\psi''(\mu), \nabla \times \psi(\mu) - \mathbf{u}_{\text{target}}\rangle + \langle \nabla \times \psi''_h(\mu), \nabla \times \delta\psi(\mu)\rangle\\ &- \langle \nabla \times \psi''_h(\mu), \mathbf{u}_{\text{target}} - \tilde{\Pi}_h \mathbf{u}_{\text{target}}\rangle|\\ &+ |\langle \nabla \times \delta\psi'(\mu), \nabla \times \psi'(\mu)\rangle + \langle \nabla \times \psi'_h(\mu), \nabla \times \delta\psi'(\mu)\rangle|\\ \leq&|\delta\psi''(\mu)|_1 \|\nabla \times \psi(\mu) - \mathbf{u}_{\text{target}}\|_{L^2(\Omega)} + |\psi''_h(\mu)|_1 |\delta\psi(\mu)|_1\\ &+ |\psi''_h(\mu)|_1 \|\mathbf{u}_{\text{target}} - \tilde{\Pi}_h \mathbf{u}_{\text{target}}\|_{L^2(\Omega)}\\ &+ |\delta\psi'(\mu)|_1 |\psi'(\mu) + \psi'_h(\mu)|_1\\ \lesssim&h^{\frac{3}{2}}.\end{aligned}$$

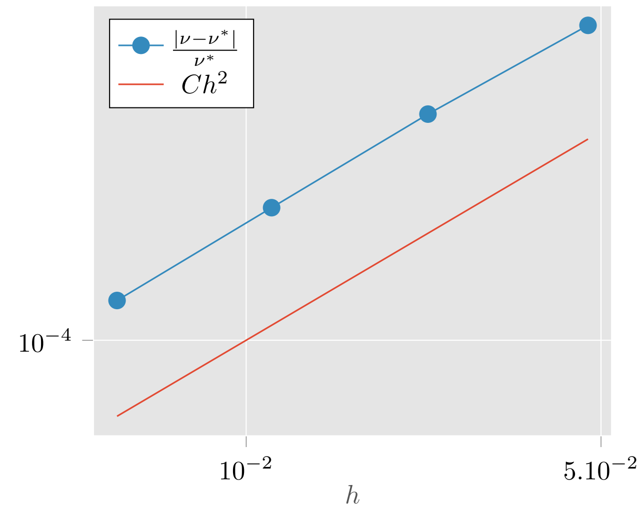

FIGURE 1. Error under mesh refinement for viscosity identification – smooth case

Therefore, conditions (15) and (16) are verified, and the proof is concluded using (18). □

Remark 3.2. *The assumption* $\|\mathbf{u}_{target}-\tilde{\Pi}_h\mathbf{u}_{target}\|_{L^2(\Omega)} = O(h^{\frac{3}{2}})$ *is reasonable, in view of theorem 1.1. Suppose that there exists* $\psi_{target} \in H^3(\Omega) \cap \mathcal{V}$ *such that* $\mathbf{u}_{target} = \nabla \times \psi_{target}$*, which is equivalent to the assumption that* $\nabla \cdot \mathbf{u}_{target} = 0$*. In this case,* $\|\mathbf{u}_{target} - \tilde{\Pi}_h\mathbf{u}_{target}\|_{L^2(\Omega)} = |\psi_{target} - \Pi_h\psi_{target}|_1 = O(h^2)$*.*

4. Numerical examples

We now show two numerical experiments in order to test the conclusion of theorem 3.2.

4.1. Smooth example

We solve the optimization problem (6) using fixed point iterations in order to solve the weak formulation (2) on the following data: $\Omega = (0,1)^2$, $\mathbf{u}_{\text{target}} = \nabla \times \psi_{\text{target}}$ where $\psi_{\text{target}}(x,y) = x^2(1-x)^2y^2(1-y)^2$. The source term f is defined through the strong formulation $f = -\nu^*\Delta^2\psi_{\text{target}} + [\Delta\psi_{\text{target}}, \psi_{\text{target}}]$, where $\nu^* = \frac{1}{100}$. Thus, an optimal solution of (6) is ν^* with optimal cost 0.

As shown in fig. 1, we retrieve a convergence of order 2, which is better than expected in theorem 3.2. This could be explained by the convergence of the discrete stream solution, which is better than the one expected in theorem 1.2 as shown in fig. 2. One can see how the convergence of the discrete stream solution influences the convergence of ν_h in the proof of theorem 3.2. Overall, the error is bounded by $|\psi(\nu) - \psi_h(\nu)|_1$ (or the norm of the derivative). Thus, if $|\psi(\nu) - \psi_h(\nu)|_1$ actually converges at a faster rate than the expected $h^{\frac{3}{2}}$ as $h \to 0$, ν_h will also converge faster. In our framework, it seems that the stream function converges at order 2 with respect to h, which was also observed in further numerical experiments in [14, section 4.3.1].

4.2. H^4 example

In order to see if the enhanced convergence does not come from the extra regularity of the previous example, we build a new example with the lower regularity. We therefore test numerically the conclusion of theorem 3.2 with an $H^4(\Omega)$ target example. On the same domain $\Omega = (0,1)^2$, we define $\mathbf{u}_{\text{target}} = \nabla \times \psi_{\text{target}}$ with

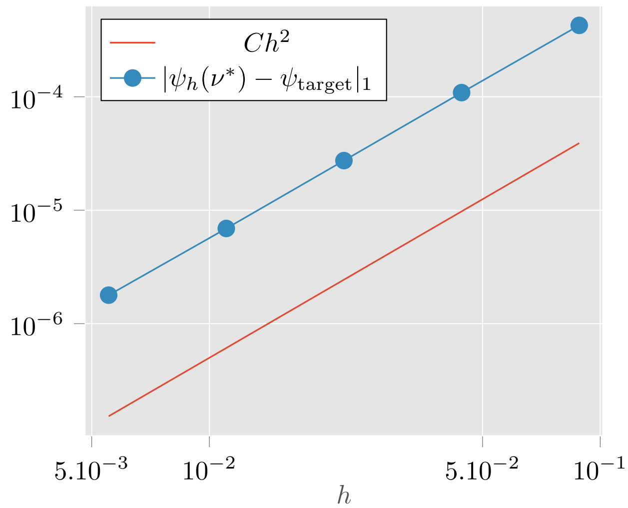

FIGURE 2. Order 2 convergence of $\psi_h(\nu^*)$ towards ψ_{target} with respect to h in H^1 semi-norm – smooth case.

$\psi_{\text{target}}(x,y) = 100\mathcal{P}(x)\mathcal{P}(y)$, where

$$\mathcal{P}(x) = \begin{cases} x(x^4 - \frac{17}{8}x^3 + \frac{17}{8}x^2 - \frac{7}{8}x) & \text{if } x < 0.5 \\ (1-x)(x^4 - \frac{7}{8}x^3 - \frac{1}{8}x) & \text{otherwise.} \end{cases}$$

One can check that $\mathcal{P}(x) \in H^4([0,1])$ but not in $H^5([0,1])$, and we can also check that $\mathcal{P}$ is defined such that $\psi_{\text{target}} \in \mathcal{V}$. The source term f is once again defined through the strong formulation $f = -\nu^*\Delta^2\psi_{\text{target}} + [\Delta\psi_{\text{target}}, \psi_{\text{target}}]$, where $\nu^* = \frac{1}{100}$.

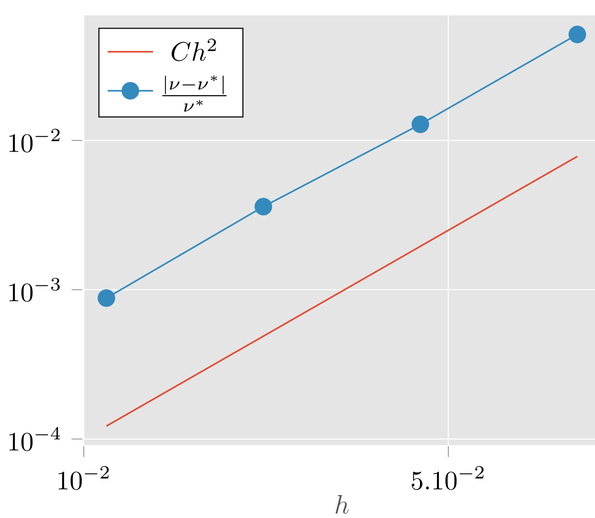

FIGURE 3. Error under mesh refinement for viscosity identification - H^4 case

As shown in fig. 3, we find once again an order 2 convergence, proving numerically that this enhanced order of convergence is not due to the regularity of the target function. As it is suggested by fig. 2, this enhanced convergence on ν is probably due to the enhanced convergence on $|\psi - \psi_h|_1$ that has not been explained yet.

5. Conclusion

We have analyzed the discretization of the viscosity identification problem in the Navier-Stokes equations. We have used the stream formulation of the Navier-Stokes equations and discretized it with Hsieh-Clough-Tocher finite elements. In this framework, we proved that the solution of the discretized problem ν_h converges to the solution of the continuous problem ν with order 3/2. In numerical experiments, we proved that this order of convergence may actually be enhanced, and discussed of how it is linked with an enhanced order of convergence of the discrete solution. This is a topic for future research.

Appendix A. Proof of convergence for $\psi_h''(\nu)$

Proof of proposition 3.3 for $\psi''(\nu)$ Define $\bar{\psi}_h''(\nu) \in \mathcal{V}_h$ as the solution of:

$$\begin{aligned}\nu a_0(\bar{\psi}_h''(\nu), \chi) &+ a_1(\bar{\psi}_h''(\nu), \psi(\nu), \chi) + a_1(\psi(\nu), \bar{\psi}_h''(\nu), \chi) \\ &= -2(a_0(\psi'(\nu), \chi) + a_1(\psi'(\nu), \psi'(\nu), \chi)), \ \forall \chi_h \in \mathcal{V}_h.\end{aligned}$$

We decompose the error: $\psi''(\nu) - \psi_h''(\nu) = \psi''(\nu) - \bar{\psi}_h''(\nu) + \bar{\psi}_h''(\nu) - \psi_h''(\nu)$.

- Let us focus first on $\delta\psi'' = \bar{\psi}_h''(\nu) - \psi_h''(\nu)$. Denote $\delta\psi = \psi(\nu) - \psi_h(\nu)$ et $\delta\psi' = \psi'(\nu) - \psi_h'(\nu)$. We have the following relation: $\forall \chi_h \in \mathcal{V}_h$

$$\begin{aligned}\nu a_0(\delta\psi'', \chi_h) + a_1(\delta\psi'', \psi(\nu), \chi_h) + a_1(\psi_h(\nu), \delta\psi'', \chi_h) = \\ - 2a_0(\delta\psi', \chi_h) - 2a_1(\delta\psi', \psi'(\nu), \chi_h) - 2a_1(\psi_h'(\nu), \delta\psi', \chi_h) \\ - a_1(\psi_h''(\nu), \delta\psi, \chi_h) - a_1(\delta\psi, \bar{\psi}_h''(\nu), \chi_h).\end{aligned}$$

  Define $\chi_h = \delta\psi''$, we now get the following estimate:

$$\begin{aligned}(\nu_{\min} - \Gamma_1|\psi(\nu)|_2)|\delta\psi''|_2^2 \leq & 2(1 + \Gamma_1|\psi'(\nu)|_2 + \Gamma_1|\psi_h'(\nu)|_2)|\delta\psi'|_2|\delta\psi''|_2 \\ & + 2\Gamma_1|\psi_h''(\nu)|_2|\delta\psi|_2|\delta\psi''|_2.\end{aligned}$$

  Using now the results on $|\delta\psi|_2$ and $|\delta\psi'|_2$, one shows that

$$|\delta\psi''|_2 \lesssim |\delta\psi|_2 + |\delta\psi'|_2 \lesssim h.$$

- We now focus on $\psi''(\nu) - \bar{\psi}_h''(\nu)$. We have the following equality: $\forall \chi_h \in \mathcal{V}_h$

$$\begin{aligned}\mathcal{L}(\psi''(\nu) - \bar{\psi}_h''(\nu), \chi_h) =& \nu a_0(\psi''(\nu) - \bar{\psi}_h''(\nu), \chi_h) \\ & + a_1(\psi''(\nu) - \bar{\psi}_h''(\nu), \psi(\nu), \chi_h) \\ & + a_1(\psi(\nu), \psi''(\nu) - \bar{\psi}_h''(\nu), \chi_h) \\ =& 0.\end{aligned}$$

  Thus, for all $\chi_h \in \mathcal{V}_h$:

$$\mathcal{L}(\psi''(\nu) - \bar{\psi}_h''(\nu), \psi''(\nu) - \bar{\psi}_h''(\nu)) = \mathcal{L}(\psi''(\nu) - \bar{\psi}_h''(\nu), \psi''(\nu) - \chi_h).$$

This relation implies that:

$$(\nu_{\min} - \Gamma_1|\psi(\nu)|_2)|\psi''(\nu) - \bar{\psi}_h''(\nu)|_2^2 \leq (\nu + 2\Gamma_1|\psi(\nu)|_2) \\ |\psi''(\nu) - \bar{\psi}_h''(\nu)|_2|\psi''(\nu) - \chi_h|_2.$$

Using the bounds on ν and $|\psi|_2$, and using theorem 1.1, one proves that:

$$|\psi''(\nu) - \bar{\psi}_h''(\nu)|_2 \lesssim \inf_{\chi_h \in \mathcal{V}_h} |\psi''(\nu) - \chi_h|_2 \lesssim h|\psi''(\nu)|_3.$$

Proof of proposition 3.4 for $\psi''(\nu)$ Denote $\delta\psi' = \psi'(\nu) - \psi_h'(\nu)$. Choosing $\chi = \psi''(\nu) - \psi_h''(\nu)$ in (5):

$$\langle g, \psi''(\nu) - \psi_h''(\nu)\rangle = \overline{\mathcal{L}}(\zeta - \zeta_h, \psi''(\nu) - \psi_h''(\nu)) + \overline{\mathcal{L}}(\zeta_h, \psi''(\nu) - \psi_h''(\nu)).$$

However, we can prove that

$$\begin{aligned}\overline{\mathcal{L}}(\zeta_h, \psi''(\nu) - \psi_h''(\nu)) = & -2(a_0(\delta\psi', \zeta_h) + a_1(\psi_h'(\nu), \delta\psi', \zeta_h) + a_1(\delta\psi', \psi'(\nu), \zeta_h)) \\ & - (a_1(\delta\psi, \psi''(\nu), \zeta_h) + a_1(\psi_h'', \delta\psi, \zeta_h)).\end{aligned}$$

Therefore, we have the following equality:

$$\begin{aligned}\langle g, \psi''(\nu) - \psi_h''(\nu)\rangle = & \overline{\mathcal{L}}(\zeta - \zeta_h, \psi''(\nu) - \psi_h''(\nu)) \\ & + 2(a_0(\delta\psi', \zeta - \zeta_h) + a_1(\psi_h'(\nu), \delta\psi', \zeta - \zeta_h) \\ & + a_1(\delta\psi', \psi'(\nu), \zeta - \zeta_h)) \\ & + a_1(\delta\psi, \psi''(\nu), \zeta - \zeta_h) + a_1(\psi_h'', \delta\psi, \zeta - \zeta_h) \\ & - 2(a_0(\delta\psi', \zeta) + a_1(\psi_h'(\nu), \delta\psi', \zeta) + a_1(\delta\psi', \psi'(\nu), \zeta)) \\ & - (a_1(\delta\psi, \psi''(\nu), \zeta) + a_1(\psi_h''(\nu), \delta\psi, \zeta)).\end{aligned}$$

Using integration by parts and Sobolev inclusions, one shows that:

$$a_0(\delta\psi', \zeta) \lesssim \|\delta\psi'\|_{L^2(\Omega)}|\zeta|_4,$$

$$\begin{aligned}a_1(\psi_h'(\nu), \delta\psi', \zeta) \leq & \|\delta\psi'\|_{L^2(\Omega)}\|\zeta\|_{W^{2,4}(\Omega)}\|\psi_h'(\nu)\|_{W^{3,4}(\Omega)} \\ \lesssim & \|\delta\psi'\|_{L^2(\Omega)}|\zeta|_3|\psi_h'(\nu)|_4, \\ a_1(\psi_h''(\nu), \delta\psi, \zeta) \lesssim & \|\delta\psi\|_{L^2(\Omega)}|\zeta|_3|\psi_h''(\nu)|_4 \\ a_1(\delta\psi', \psi'(\nu), \zeta) \leq & \|\delta\psi'\|_{L^2(\Omega)}\|\zeta\|_{W^{3,4}(\Omega)}\|\psi'(\nu)\|_{W^{3,4}(\Omega)} \\ \lesssim & \|\delta\psi'\|_{L^2(\Omega)}|\zeta|_4|\psi'(\nu)|_4, \\ a_1(\delta\psi, \psi''(\nu), \zeta) \lesssim & \|\delta\psi\|_{L^2(\Omega)}|\zeta|_4|\psi''(\nu)|_4.\end{aligned}$$

Using the hypothesis $|\zeta|_4 \lesssim \|g\|_{L^2(\Omega)}$, we get:

$$\begin{aligned}\langle g, \psi''(\nu) - \psi_h''(\nu)\rangle \lesssim & (\nu + 2\Gamma_1|\psi(\nu)|_2)|\zeta - \zeta_h|_2|\psi''(\nu) - \psi_h''(\nu)|_2 \\ & + 2(1 + \Gamma_1(|\psi_h'(\nu)|_2 + |\psi'(\nu)|_2))|\zeta - \zeta_h|_2|\delta\psi'|_2 \\ & + \Gamma_1(|\psi_h''(\nu)|_2 + |\psi''(\nu)|_2)|\zeta - \zeta_h|_2|\delta\psi|_2 \\ & + (1 + |\psi'(\nu)|_4 + |\psi_h'(\nu)|_4)\|\delta\psi'\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)} \\ & + (|\psi''(\nu)|_4 + |\psi_h''(\nu)|_4)\|\delta\psi\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)}.\end{aligned}$$

Using the interpolation error between ζ and ζ_h (see theorem 1.1) and the error on $|\delta\psi|_2$, $|\delta\psi'|_2$ and $|\psi''(\nu) - \psi_h''(\nu)|_2$, we get:

$$\langle g, \psi''(\nu) - \psi_h''(\nu)\rangle \lesssim h^2 \|g\|_{L^2(\Omega)}.$$

Choosing now $g = \Delta(\psi''(\nu) - \psi_h''(\nu))$, we get:

$$|\psi''(\nu) - \psi_h''(\nu)|_1^2 = \langle g, \psi''(\nu) - \psi_h''(\nu)\rangle \lesssim h^2 |\psi''(\nu) - \psi_h''(\nu)|_2 \lesssim h^3,$$

which proves that $|\psi''(\nu) - \psi_h''(\nu)|_1 \lesssim h^{\frac{3}{2}}$. Choosing now $g = \psi''(\nu) - \psi_h''(\nu)$, we have that $\|\psi''(\nu) - \psi_h''(\nu)\|_{L^2(\Omega)} \lesssim h^2$.

The author would like to thank Didier Auroux for his idea of testing the influence of the regularity of the target on the order of convergence, Hervé Guillard for providing his code implementing HCT elements, and Florence Marcotte for the proofreading and the discussion.

References

[1] Abderrahim Azouani, Eric Olson, and Edriss S Titi. Continuous data assimilation using general interpolant observables. *Journal of Nonlinear Science*, 24:277–304, 2014.

[2] Li Bingjie and Liu Sanyang. On gradient-type optimization method utilizing mixed finite element approximation for optimal boundary control problem governed by bi-harmonic equation. *Applied mathematics and computation*, 186(2):1429–1440, 2007.

[3] Animikh Biswas and Joshua Hudson. Determining the viscosity of the navier–stokes equations from observations of finitely many modes. *Inverse Problems*, 39(12):125012, 2023.

[4] Heribert Blum, Rolf Rannacher, and Rolf Leis. On the boundary value problem of the biharmonic operator on domains with angular corners. *Mathematical Methods in the Applied Sciences*, 2(4):556–581, 1980.

[5] Laurent Bourgeois. A remark on lipschitz stability for inverse problems. *Comptes Rendus Mathematique*, 351(5-6):187–190, 2013.

[6] Weidong Cao and Danping Yang. Ciarlet–raviart mixed finite element approximation for an optimal control problem governed by the first bi-harmonic equation. *Journal of computational and applied mathematics*, 233(2):372–388, 2009.

[7] Elizabeth Carlson, Joshua Hudson, and Adam Larios. Parameter recovery for the 2 dimensional Navier–Stokes equations via continuous data assimilation. *SIAM Journal on Scientific Computing*, 42(1):A250–A270, 2020.

[8] Eduardo Casas and Fredi Tröltzsch. Second order optimality conditions and their role in PDE control. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 117(1):3–44, 2015.

[9] ME Cayco and RA Nicolaides. Finite element technique for optimal pressure recovery from stream function formulation of viscous flows. *Mathematics of computation*, 46(174):371–377, 1986.

[10] Philippe G Ciarlet. Interpolation error estimates for the reduced Hsieh-Clough-Tocher triangle. *Mathematics of Computation*, 32(142):335–344, 1978.

[11] Pierre-Henri Cocquet, Michaël Rakotobe, Delphine Ramalingom, and Alain Bastide. Error analysis for the finite element approximation of the Darcy–Brinkman–Forchheimer model for porous media with mixed boundary conditions. *Journal of Computational and Applied Mathematics*, 381:113008, 2021.

[12] Klaus Deckelnick and Michael Hinze. Error estimates in space and time for tracking-type control of the instationary stokes system. In *Control and Estimation of Distributed Parameter Systems: International Conference in Maria Trost (Austria), July 15–21, 2001*, pages 87–103. Springer, 2003.

[13] Shaohong Du and Xiaoxia He. Finite element approximation to optimal Dirichlet boundary control problem: A priori and a posteriori error estimates. *Computers & Mathematics with Applications*, 131:14–25, 2023.

[14] Ali Aboudou Elarif. *Approximation par éléments finis C1 des modèles magnétohydrodynamiques pour les plasmas de fusion.* PhD thesis, 2020. Université Côte d'Azur 2020.

[15] Jishan Fan, Michele Di Cristo, Yu Jiang, and Gen Nakamura. Inverse viscosity problem for the navier–stokes equation. *Journal of mathematical analysis and applications*, 365(2):750–757, 2010.

[16] Stefan Frei, Rolf Rannacher, and Winnifried Wollner. A priori error estimates for the finite element discretization of optimal distributed control problems governed by the biharmonic operator. *Calcolo*, 50:165–193, 2013.

[17] Vivette Girault and Pierre-Arnaud Raviart. *Finite element methods for Navier-Stokes equations: theory and algorithms*, volume 5. Springer Science & Business Media, 2012.

[18] Thirupathi Gudi, Neela Nataraj, and Kamana Porwal. An interior penalty method for distributed optimal control problems governed by the biharmonic operator. *Computers & Mathematics with Applications*, 68(12):2205–2221, 2014.

[19] Max D Gunzburger and L Steven Hou. Finite-dimensional approximation of a class of constrained nonlinear optimal control problems. *SIAM journal on control and optimization*, 34(3):1001–1043, 1996.

[20] Michael Hinze, René Pinnau, Michael Ulbrich, and Stefan Ulbrich. *Optimization with PDE constraints*, volume 23. Springer Science & Business Media, 2008.

[21] O Yu Imanuvilov and M Yamamoto. Global uniqueness in inverse boundary value problems for the navier–stokes equations and lamé system in two dimensions. *Inverse Problems*, 31(3):035004, 2015.

[22] Bangti Jin and Zhi Zhou. Error analysis of finite element approximations of diffusion coefficient identification for elliptic and parabolic problems. *SIAM Journal on Numerical Analysis*, 59(1):119–142, 2021.

[23] Matti Lassas, Tony Liimatainen, Yi-Hsuan Lin, and Mikko Salo. Inverse problems for elliptic equations with power type nonlinearities. *Journal de mathématiques pures et appliquées*, 145:44–82, 2021.

[24] Xiaosheng Li and Jenn-Nan Wang. Determination of viscosity in the stationary navier–stokes equations. *Journal of Differential Equations*, 242(1):24–39, 2007.

[25] Robert Luce and Sylvie Perez. Parameter identification for an elliptic partial differential equation with distributed noisy data. *Inverse Problems*, 15(1):291, 1999.

[26] Gouranga Mallik, Neela Nataraj, and Jean-Pierre Raymond. Error estimates for the numerical approximation of a distributed optimal control problem governed by the von kármán equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 52(3):1137–1172, 2018.

[27] Vincent R Martinez. Convergence analysis of a viscosity parameter recovery algorithm for the 2d Navier–Stokes equations. *Nonlinearity*, 35(5):2241, 2022.

[28] Dominik Meidner and Boris Vexler. Adaptive space-time finite element methods for parabolic optimization problems. *SIAM Journal on Control and Optimization*, 46(1):116–142, 2007.

[29] Pedro Merino, Fredi Tröltzsch, and Boris Vexler. Error estimates for the finite element approximation of a semilinear elliptic control problem with state constraints and finite dimensional control space. *ESAIM: Mathematical Modelling and Numerical Analysis*, 44(1):167–188, 2010.

[30] Gilbert Peralta and John Sebastian Simon. Optimal control for the Navier–Stokes Equation with time delay in the convection: Analysis and finite element approximations. *Journal of Mathematical Fluid Mechanics*, 23(3):56, 2021.

[31] Rolf Rannacher and Boris Vexler. A priori error estimates for the finite element discretization of elliptic parameter identification problems with pointwise measurements. *SIAM Journal on Control and Optimization*, 44(5):1844–1863, 2005.

[32] Zhen-Zhen Tao, Bing Sun, and Hai-Feng Niu. Galerkin spectral approximation for optimal control problem of a fourth-order equation with l 2-norm control constraint. *International Journal of Computer Mathematics*, 99(7):1344–1366, 2022.

[33] Boris Vexler. *Adaptive finite element methods for parameter identification problems*. PhD thesis, 2004.