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PARAMETERS IDENTIFICATION IN 2D REDUCED MHD EQUATIONS USING PARTIAL FOURIER MODES OBSERVATIONS. *

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Abstract. We are interested in the problem of identifying the viscosity and resistivity in a 2D MHD equations based on observations of low Fourier modes only. In this regard, we analyse a system identification method called nudging in the case where these parameters are only approximately known. The inverse problem of identifying these parameters given partial observations is posed as an optimization problem, and we prove uniqueness of the solution and a lipschitz property. Eventually, we derive a numerical method in order to retrieve the viscosity and resistivity that we test numerically.

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1. Introduction

It is usual for many physical systems to be modeled with differential equations using different parameters ; for instance viscosity, thermal resistivity, or diffusivity. A precise value of these parameters, along with a precise knowledge of the state of the physical system at some initial time, are necessary in order to produce accurate or practical simulations of these systems. However, these values are often unknown, or not precisely. This is the case for instance for some systems dealing with nuclear fusion, which are sometimes described using a magnetohydrodynamic (MHD) model. In this model, two scalar parameters appear ; namely, the kinematic fluid viscosity and the magnetic diffusivity. In practical framework, these values need to be adjusted to the observations we have at hand. This is where the techniques of *data assimilation* become useful.

The data assimilation techniques are a family of techniques developed in order to include information from, for instance, measurements or observations, into simulations in order to increase their accuracy. These techniques are often based on the Kalman filter, which is a linear quadratic estimator with Gaussian measurement noise. The main idea is to design a stochastic time-varying linear model for the dynamic and the observations with some additional noise that has a centered Gaussian distribution with unknown standard deviation. The goal is then to infer the *true* dynamic and the additional noise based on the *polluted* observations. One of its drawback is that the addition of the Gaussian noise is often not enough to compensate the unmodeled dynamics. Nonetheless, these techniques were applied and extended for different problems, including for nonlinearity, leading to powerful smoothing (Kalman-Bucy smoothers, MCMC, ...) or filtering algorithms (3DVAR, Extended and Ensemble

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Kalman Filters, ...). For more on Kalman Filters, see [21, 23]. We also mention the paper by Blömker et al. [7] who applied the 3DVAR algorithm to the Navier-Stokes equations, and [1] for an iterative scheme for a parameter and initial condition identification related to these methods.

In the last decade, a technique designed by Azouani, Olson, and Titi [3] proposes a new approach to data assimilation (in the context of the nonlinear Navier-Stokes equations), usually called in the literature the AOT algorithm or Continuous Data Assimilation (CDA). Their idea is based on results in the control theory community, and has several links with state observers such as Luenberger observers or the backstepping methods ; see [30, Chapter 7] and [22]. The main ingredient of the AOT algorithm is the observation feedback penalization, which adds an extra source term, function of the gap between the simulated model and the observation. This penalization term *forces* the simulated system to converge exponentially in time to the observed variables. This method also has the advantages of possibly exploiting only sparse observations of the state variables, and can be used online for real-time assimilation. The work by Azouani, Olson, and Titi focused on observations obtained noise-free and continuously in time, but these two drawbacks have since been overcomed [4, 17]. This approach has been successfully used for other models than the Navier–Stokes equations, such as the 2D Bénard convection equations [2] or the 2D MHD equations [6, 19]. A flaw of this method is that only the initial state condition is unknown, but all other parameters are supposed to be exactly known. The interested reader will find more details in [12, 14, 20, 26, 27], among others.

More recently, the AOT algorithm have been extended to identify unknown parameters. Carlson et al. in [10] prove that the AOT algorithm applied to the Navier–Stokes equations is stable with respect to a change of viscosity parameter, and proposes two algorithms to dynamically recover the true viscosity that produced the observations. This approach has then been proved to converge in [5, 25]. This approach has also been applied to chaotic ODEs, see [11].

Our goal in this paper is to adapt the results of [5] to the 2D MHD equations in order to identify the kinematic fluid viscosity and the magnetic diffusivity. This problem has for instance practical interests for the simulation of a fusion nuclear reaction in a tokamak [34], where only partial sparse data can be observed inside the chamber.

This paper is organized as follows. In section 2, we recall some background results concerning the AOT algorithm and the MHD equations. In section 3, we define the determining map and prove some estimates on this map in term of source terms. These estimates are then used in section 4, where we analyse the parameter identification inverse problem, posed as an optimization problem. Finally, in section 5, we prove the convergence of an iterative scheme in order to recover the unknown parameters and expose some numerical experiments illustrating the results. For ease of presentation of these results, the longest proofs are gathered in the appendix A.

Notations For some domain $\Omega \subset \mathbb{R}^2$, we denote $|\cdot|$ the norm on $L^2(\Omega)$, $\|\cdot\|$ the norm on $H^1(\Omega)$. For some normed space $\mathbb{X}$, we denote its norm $\|\cdot\|_{\mathbb{X}}$, and $C_b(\mathbb{X})$ the set of continuous bounded mappings from $\mathbb{R}$ to $\mathbb{X}$. It is a normed space with norm $\|\phi\|_{C_b(\mathbb{X})} = \sup_{t\in\mathbb{R}} \|\phi(t)\|_{\mathbb{X}}$.

2. Preliminaries

Background on Data Assimilation We expose the main ideas we are going to use for the 2D MHD equations. The idea of Azouani, Olson and Titi in [3] is based on a feedback control built on the observations. Suppose we study a dynamical system of the form $\frac{d}{dt}Y = F(Y)$ subject to an initial condition $Y(0) = Y_0$ which is unknown. Since the solution $t \mapsto Y(t)$ depends continuously on its initial condition, an approximation $\tilde{Y}_0$ of Y_0 based on observations ensures that the solution $t \mapsto \tilde{Y}(t)$ such that $\tilde{Y}(0) = \tilde{Y}_0$ stays close to Y. However, this approach produces an approximation $\tilde{Y}$ which may be valid only for a short amount of time ; asymptotically in time, the error between $\tilde{Y}$ and Y may grow dramatically. Also, it should be noted that measurements are often sparse or noisy, so a good approximation of Y_0 can be hardly guaranteed.

Instead of trying to approximate the initial condition, we will continuously feed back into the equations the observations we have, and which are supposed to be given continuously. This way, the approximation could be wrong at the beginning, but will become better as time goes on. In the article of Azouani et al., we suppose we are given an interpolant operator I_h of the (sparse) data with parameter h (which can be seen as the mesh size for instance). Given that the true value of the (partial) observation of Y at time t is $I_h(Y(t))$, we approximate Y by finding $\widetilde{Y}$, solution of the equation:

$$\frac{d}{dt}\widetilde{Y} = F(\widetilde{Y}) + \mu(I_h(Y) - I_h(\widetilde{Y})). \tag{1}$$

Here, μ is a tuning parameter which will have to be chosen big enough so that the identification process converges. In the present work, we will suppose that $I_h = P_N$, the Fourier modal projection operator, which is defined as:

$$\mathcal{F}(P_N\phi)(\mathbf{k}) = \begin{cases} \mathcal{F}(\phi)(\mathbf{k}) & \text{if } |\mathbf{k}| < N, \\ 0 & \text{otherwise,} \end{cases} \quad \forall \mathbf{k} \in \mathbb{Z}^2,$$

where $\mathcal{F}(\cdot)$ denotes the Fourier transform. The parameter N is seen as the inverse of h : $N = h^{-1}$. One easily proves that P_N has the following property:

$$\forall u \in H^1,\ |u - P_N(u)| \leq \frac{c_P}{N}\|u\|, \tag{2}$$

for some $c_P > 0$. Many other interpolation operators respect this condition, such as finite volume element or nodal interpolation ; see for instance [3, 18, 24].

2D MHD equations We are interested in the reduced MHD equations, reading:

$$\begin{aligned} &\partial_t \mathbf{u} - \mathrm{Re}^{-1}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{b}\cdot\nabla)\mathbf{b} + \nabla p = \tilde{f} \text{ in } \Omega, \\ &\partial_t \mathbf{b} - \mathrm{Rm}^{-1}\Delta\mathbf{b} + (\mathbf{u}\cdot\nabla)\mathbf{b} - (\mathbf{b}\cdot\nabla)\mathbf{u} = \tilde{g} \text{ in } \Omega, \\ &\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{b} = 0, \\ &\mathbf{u}(0),\ \mathbf{b}(0) \text{ given.} \end{aligned} \tag{3}$$

where $f, g \in L^2(\Omega)$ are a given source terms, $\nabla\cdot$ denote the divergence, $\mathbf{u}$ is the velocity vector field, $\mathbf{b}$ the magnetic field, p the pressure, Re the unknown fluid Reynolds number, and Rm the unknown magnetic Reynolds number. All variables and parameters are non-dimensionalized. For ease of presentation, we choose $\Omega = \mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$, the 2 dimensional torus, which is an open, bounded and connected domain. The subsequent results can be generalized to more general domains in 2 dimensions, as long as we keep the periodical boundary conditions.

We assume, without loss of generality, that Re $<$ Rm, and denote the Elsässer variables [13] by $\mathbf{v} = \mathbf{u} + \mathbf{b}$ and $\mathbf{w} = \mathbf{u} - \mathbf{b}$ (if Re $\geq$ Rm then we would denote $\mathbf{w} = \mathbf{b} - \mathbf{u}$ and proceed similarly). The equations verified by $\mathbf{v}$ and $\mathbf{w}$ then becomes:

$$\begin{aligned} &\partial_t \mathbf{v} - \alpha\Delta\mathbf{v} - \beta\Delta\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{v} + \nabla p = f \text{ in } \Omega, \\ &\partial_t \mathbf{w} - \alpha\Delta\mathbf{w} - \beta\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{w} + \nabla p = g \text{ in } \Omega, \\ &\nabla\cdot\mathbf{v} = \nabla\cdot\mathbf{w} = 0, \\ &\mathbf{v}(0) = \mathbf{u}(0) + \mathbf{b}(0),\ \mathbf{w}(0) = \mathbf{u}(0) - \mathbf{b}(0). \end{aligned} \tag{4}$$

where $f = \tilde{f} + \tilde{g}$, $g = \tilde{f} - \tilde{g}$, $\alpha = \frac{1}{2}(\mathrm{Re}^{-1} + \mathrm{Rm}^{-1})$ and $\beta = \frac{1}{2}(\mathrm{Re}^{-1} - \mathrm{Rm}^{-1})$. It will be important to note that, due to our assumptions, $\alpha - \beta = \mathrm{Rm}^{-1} > 0$ and that $\alpha > 0$ and $\beta \geq 0$.

As is customary, we define the space $\mathcal{V} := \{\mathbf{u} \in C^\infty(\mathbb{R}^2/\mathbb{Z}^2;\mathbb{R}^2) \mid \nabla\cdot\mathbf{u} = 0, \int_\Omega \mathbf{u} = 0\}$ and, subsequently, the spaces $H := \overline{\mathcal{V}}$ in $L^2(\Omega;\mathbb{R}^2)$ and $V := \overline{\mathcal{V}}$ in $H^1(\Omega;\mathbb{R}^2)$. H and V are Hilbert spaces with the inner product defined as

$$\langle \mathbf{u}, \mathbf{v}\rangle = \int_\Omega \mathbf{u}\cdot\mathbf{v},\ \langle\langle \mathbf{u}, \mathbf{v}\rangle\rangle = \sum_{i,j=1}^{2}\int_\Omega \frac{\partial \mathbf{u}_i}{\partial x_j}\frac{\partial \mathbf{v}_i}{\partial x_j},$$

with corresponding norms $|\mathbf{u}| = \sqrt{\langle \mathbf{u}, \mathbf{u}\rangle}$ and $\|\mathbf{u}\| = \sqrt{\langle\langle \mathbf{u}, \mathbf{u}\rangle\rangle}$. We denote the Leray projection $P_\sigma : L^2(\Omega) \to H$ defined by $P_\sigma \mathbf{u} = \mathbf{u} - \nabla\Delta^{-1}(\nabla \cdot \mathbf{u})$ (see [8]). P_σ is the orthogonal projection of a vector field onto its divergence-free part. We will apply the Leray operator to (4). We define the Stokes operator $\mathcal{A}$ and the bilinear term $\mathcal{B} : V \times V \to V^*$ as the continuous extensions of the operators $\mathcal{A}$ and $\mathcal{B}$ defined on $\mathcal{V} \times \mathcal{V}$ as

$$\mathcal{A}u = -P_\sigma(\Delta \mathbf{u}) \text{ and } \mathcal{B}(\mathbf{u}, \mathbf{v}) = P_\sigma((\mathbf{u} \cdot \nabla)\mathbf{v}),$$

and we define the domain of $\mathcal{A}$ to be $D(\mathcal{A}) = \{\mathbf{u} \in V : \mathcal{A}\mathbf{u} \in H\}$. In order to simplify the notations, we will suppose that $\nabla \cdot f = \nabla \cdot g = 0$; if not, replace in subsequent equations f and g and $P_\sigma(f)$ and $P_\sigma(g)$. Thus, once P_σ is applied to (4), it becomes:

$$\begin{aligned} &\partial_t \mathbf{v} + \alpha\mathcal{A}\mathbf{v} + \beta\mathcal{A}\mathbf{w} + \mathcal{B}(\mathbf{w}, \mathbf{v}) = f \text{ in } \Omega, \\ &\partial_t \mathbf{w} + \alpha\mathcal{A}\mathbf{w} + \beta\mathcal{A}\mathbf{v} + \mathcal{B}(\mathbf{v}, \mathbf{w}) = g \text{ in } \Omega, \\ &\mathbf{v}(0) = \mathbf{u}(0) + \mathbf{b}(0),\ \mathbf{w}(0) = \mathbf{u}(0) - \mathbf{b}(0). \end{aligned} \tag{5}$$

The operator $\mathcal{A}$ is an unbounded self-adjoint positive operator on H [31, chapter 2]. As stated in [33], the norm $|\mathcal{A}\mathbf{u}|$ on $D(\mathcal{A})$ is equivalent to the norm induced by $(H^2(\Omega))^2$. The spectrum of $\mathcal{A}$ consists in an infinite countable sequence of eigenvalues with lowest eigenvalue $\lambda_1 = c_P = 4\pi^2$ in the case where $\Omega = \mathbb{T}$. The natural extension of the operator $\mathcal{A}$ to V defines an isomorphism onto its dual V', and the following holds:

$$\langle\langle \mathbf{u}, \mathbf{v}\rangle\rangle = \sum_{i,j=1}^{2} \int_\Omega \partial_{x_j}\mathbf{u}_i \partial_{x_j}\mathbf{v}_i = \langle \mathcal{A}\mathbf{u}, \mathbf{v}\rangle,\ \forall \mathbf{u}, \mathbf{v} \in V.$$

It also implies:

$$\|\mathbf{v}\| \geq \lambda_1^{\frac{1}{2}}|\mathbf{v}|,\ \forall \mathbf{v} \in V,$$

$$|\mathcal{A}\mathbf{v}| \geq \lambda_1^{\frac{1}{2}}\|\mathbf{v}\|,\ \forall \mathbf{v} \in D(\mathcal{A}),$$

The bilinear operator $\mathcal{B}$ has several properties that we recall here. Proof of these results can be found in [32]. First of all, $\mathcal{B}$ has the following property:

$$\langle \mathcal{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle = -\langle \mathcal{B}(\mathbf{u}, \mathbf{w}), \mathbf{v}\rangle,\ \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V \tag{6}$$

This implies in particular that:

$$\langle \mathcal{B}(\mathbf{u}, \mathbf{v}), \mathbf{v}\rangle = 0,\ \forall \mathbf{u}, \mathbf{v} \in V \tag{7}$$

Moreover, we have the following bounds:

$$\langle \mathcal{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle \leq c_2^{\mathcal{B}}|\mathbf{u}|^{\frac{1}{2}}\|\mathbf{u}\|^{\frac{1}{2}}\|\mathbf{v}\||\mathbf{w}|^{\frac{1}{2}}\|\mathbf{w}\|^{\frac{1}{2}},\ \mathbf{u}, \mathbf{v}, \mathbf{w} \in V, \tag{8}$$

$$\langle \mathcal{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle \leq c_5^{\mathcal{B}}|\mathbf{u}|^{\frac{1}{2}}\|\mathbf{u}\|^{\frac{1}{2}}\|\mathbf{v}\|^{\frac{1}{2}}|\mathcal{A}\mathbf{v}|^{\frac{1}{2}}|\mathbf{w}|,\ \mathbf{u} \in V, \mathbf{v} \in D(\mathcal{A}), \mathbf{w} \in H. \tag{9}$$

Attractor of the MHD equations A well known fact about the solutions $(\mathbf{v}, \mathbf{w})$ of (4) is their boundedness. More precisely, as it is shown in [29, Theorems 3.1 and 3.2], for any $T > 0$:

- If $f, g \in L^2(0, T; V')$ and $\mathbf{v}(0), \mathbf{w}(0) \in H$, then $\mathbf{v}, \mathbf{w} \in L^2(0, T; V) \cap \mathcal{C}(0, T; H)$,
- If $f, g \in L^2(0, T; H)$ and $\mathbf{v}(0), \mathbf{w}(0) \in V$, then $\mathbf{v}, \mathbf{w} \in L^2(0, T; D(\mathcal{A})) \cap L^\infty(0, T; V)$.

Furthermore, if one supposes:

$$\operatorname*{ess\,sup}_{[0,\infty)} \max\{|f'|,\ |g'|\} < \infty \text{ and } \mathbf{v}(0), \mathbf{w}(0) \in H,$$

then $\mathbf{v}, \mathbf{w} \in L^\infty((0, \infty); D(\mathcal{A}))$; see [29, Theorem 4.1].

These boundedness properties are related to the notion of absorbing set and attractors. A bounded set $\mathfrak{B} \subset H$ is called absorbing with respect to the semi-group $\{S(t)\}_{t\geq 0}$ if, for any bounded subset $B \subset H$, there exists a time $T = T(B)$ such that $S(t)B \subset \mathfrak{B}$ for all $t \geq T$. The smallest absorbing set is called the global attractor, and is defined as follows:

Definition 2.1. [28] Let $\mathfrak{B} \subset H$ be a bounded absorbing set with respect to the semi-group $\{S(t)\}_{t\geq 0}$. Then the global attractor $\mathfrak{A}$ exists as given by any of the equivalent definitions:

(1) $\mathfrak{A} = \bigcap_{t\geq 0} S(T)\mathfrak{B}$.
(2) $\mathfrak{A}$ is the biggest compact subset of H which is invariant under the action of $\{S(t)\}$, i.e. for all $t \geq 0$, $S(t)\mathfrak{A} = \mathfrak{A}$.
(3) $\mathfrak{A}$ is the smallest set that attracts all bounded sets.

The global attractor lets us also define the semi-group $\{S(t)\}$ for all $t \in \mathbb{R}$. As proved in [28, Theorem 10.6], if the semi-group is injective on $\mathcal{A}$, i.e.:

$$\text{For some } t > 0,\ S(t)\mathbf{u}_0 = S(t)\mathbf{v}_0 \in \mathfrak{A} \implies \mathbf{u}_0 = \mathbf{v}_0,$$

then every trajectory on $\mathfrak{A}$ is defined for all $t \in \mathbb{R}$, and $S(t)\mathfrak{A} = \mathfrak{A}$ for all $t \in \mathbb{R}$. Thus if the injective semi-group $\{S(t)\}$ is bounded on some time-interval, the semi-group is defined and bounded for all $t \in \mathbb{R}$.

Concerning the solution of (4), one can prove that its semi-group of solution is injective ; the proof is similar to the proof for the Navier-Stokes equations, see [28, Theorems 11.10, 12.8 and Corollary 12.9] and is partly discussed in [29, Section 4.2 and 5]. Thus, under some hypothesis on the data, one has $\mathbf{v}, \mathbf{w} \in C_b(H)$ or $\mathbf{v}, \mathbf{w} \in C_b(H) \cap L^\infty(\mathbb{R}; V)$. For ease of presentation, we will denote $C_b(V) = C_b(H) \cap L^\infty(\mathbb{R}; V)$.

Data assimilation system We now build a data assimilation system based on the idea exposed in (1). In this case, we also take into account that we only have an approximate idea of the value of the fluid and magnetic viscosities. Suppose we observe the functions $\phi^v, \phi^w \in C_b(V)$. Thus, we approximate the fields ϕ^v and ϕ^w with the solutions $\widetilde{\mathbf{v}}$ and $\widetilde{\mathbf{w}}$ of:

$$\begin{aligned} &\partial_t \widetilde{\mathbf{v}} + \widetilde{\alpha}\mathcal{A}\widetilde{\mathbf{v}} + \widetilde{\beta}\mathcal{A}\widetilde{\mathbf{w}} + \mathcal{B}(\widetilde{\mathbf{w}}, \widetilde{\mathbf{v}}) = f + \mu P_\sigma(\phi^v - P_N(\widetilde{\mathbf{v}})), \\ &\partial_t \widetilde{\mathbf{w}} + \widetilde{\alpha}\mathcal{A}\widetilde{\mathbf{w}} + \widetilde{\beta}\mathcal{A}\widetilde{\mathbf{v}} + \mathcal{B}(\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}}) = g + \mu P_\sigma(\phi^w - P_N(\widetilde{\mathbf{w}})), \\ &\widetilde{\mathbf{v}}(0) = \widetilde{\mathbf{w}}(0) = 0, \end{aligned} \tag{10}$$

where $\widetilde{\alpha}$, $\widetilde{\beta}$ are approximations of α and β. A priori estimate on the solution, existence and uniqueness of strong solutions to (10) will be the topic of section 3.

Other inequalities We now recall some inequalities which will be used throughout the proofs.

- Young's inequality : for any $p > 0, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and all $a \geq 0, b \geq 0$, $\kappa > 0$

$$ab \leq \frac{a^p}{\kappa^p p} + \frac{\kappa^q b^q}{q}. \tag{11}$$

- The continuous inclusion of $H^1(\Omega)$ into $L^4(\Omega)$, as proved by Ladyzhenskaya: there exists $c_L > 0$ such that, for all $u \in H^1(\Omega)$, $|\mathbf{u}|^2_{L^4(\Omega)} \leq c_L |\mathbf{u}| \|\mathbf{u}\|$

3. Determining map

We now define the map of viscosities and partial observations to the solution of (10). This map has been first defined in [15, 16] using only the observations, and has been extended in [5] to incorporate the viscosity in the Navier-Stokes equation.

Definition 3.1. Fix an arbitrary radius $R > 0$ for a ball $B_R(0) \subset C_b(V)$. The determining map $\mathbb{W} : (\mathbb{R}_{>0})^2 \times (B_R(0))^2 \to (C_b(V))^2$, is the mapping of viscosity, resistivity and data to the corresponding strong solution of (10) on the attractor.

As an example, $\mathbb{W}(\alpha^*, \beta^*, P_N\mathbf{v}, P_N\mathbf{w})$ returns the solution $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}})$ of (10) using $(\widetilde{\alpha}, \widetilde{\beta}) = (\alpha^*, \beta^*)$ and $(\phi^v, \phi^w) = (P_N\mathbf{v}, P_N\mathbf{w})$. We will prove that, under some sufficient conditions on N and μ, $\mathbb{W}$ is a well-defined lipschitz continuous operator. But first, we derive an a priori estimate in H and in $L^2(t-\tau, t; V)$ on the solution of (10). The proof of this result has been moved to appendix A.

Lemma 3.1. *Fix $\nu_{\min} > 0$ and suppose that:*

$$\widetilde{\alpha} - \widetilde{\beta} > \nu_{\min}, \ \mu \geq \frac{\nu_{\min} N^2}{c_P^2}.$$

For any $f, g, \phi^v, \phi^w \in C_b(V)$, and for any $N \geq 1$, if $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}}) \in C_b(H)$ are strong solutions of (10), then the following bounds are satisfied:

$$\begin{aligned} \|\widetilde{\mathbf{v}}\|^2_{C_b(H)} + \|\widetilde{\mathbf{w}}\|^2_{C_b(H)} \leq & \left(\frac{2c_P^2}{\nu_{\min}}\right)^2 (\|f\|^2_{C_b(H)} + \|g\|^2_{C_b(H)}) \\ & + 2\left(\frac{\mu c_P^2}{\nu_{\min}}\right)^2 (\|\phi^v\|^2_{C_b(H)} + \|\phi^w\|^2_{C_b(H)}) \\ = & (M_H(\phi^v, \phi^w))^2, \end{aligned} \tag{12}$$

and for all $t \in \mathbb{R}$ and all $\tau > 0$:

$$\begin{aligned} \int_{t-\tau}^t (\|\widetilde{\mathbf{v}}(s)\|^2 + \|\widetilde{\mathbf{w}}(s)\|^2) ds \leq & \left(\frac{8c_P^2}{9\nu_{\min}^2 N^2}\tau + \frac{8c_P^4}{3\nu_{\min}^3 N^4}\right) (\|f\|^2_{C_b(H)} + \|g\|^2_{C_b(H)}) \\ & + \left(\frac{32\mu^2 c_P^2}{3\nu_{\min}^2 N^2}\tau + \frac{4\mu^2 c_P^4}{3\nu_{\min}^3 N^4}\right) (\|\phi^v\|^2_{C_b(H)} + \|\phi^w\|^2_{C_b(H)}) \\ = & M_V^{L_2}(\phi^v, \phi^w; \tau)^2 \end{aligned} \tag{13}$$

Now, we derive an a priori estimate in V on the solution of (10). Note that, contrary to the analysis carried on Navier-Stokes equations in [5], the proof is more involved as we can not use the enstrophy cancelation property with the MHD equations. Thus, we need to follow arguments close to the ones given in [29] for the classical 2D MHD equations. The proof of this result is available in appendix A.

Lemma 3.2. *Fix $\nu_{\min} > 0$ and suppose that:*

$$\widetilde{\alpha} - \widetilde{\beta} > \nu_{\min}, \ \mu \geq \frac{\nu_{\min} N^2}{2c_P^2}.$$

For any $f, g, \phi^v, \phi^w \in C_b(V)$, and for any $N \geq 1$, if $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}}) \in C_b(V)$ are strong solutions of (10), then the following bounds are satisfied:

$$\begin{aligned} \|\widetilde{\mathbf{v}}\|^2_{C_b(V)} + \|\widetilde{\mathbf{w}}\|^2_{C_b(V)} \leq & \exp\left(\frac{27(c_5^B)^4}{32\nu_{\min}^5} M_H(\phi^v, \phi^w)^4\right) \left(\frac{8}{c_P^2} M_H(\phi^v, \phi^w)^2\right. \\ & + \frac{c_P^4}{2\nu_{\min}^2} (\|f\|^2_{C_b(V)} + \|g\|^2_{C_b(V)}) \\ & \left. + \frac{\mu^2 c_P^4}{2\nu_{\min}^2} (\|\phi^v\|^2_{C_b(V)} + \|\phi^w\|^2_{C_b(V)})\right) \\ = & (M_V(\phi^v, \phi^w))^2. \end{aligned} \tag{14}$$

We can now derive sufficient conditions on μ and N in order to make sure that $\mathbb{W}$ is lipschitz continuous. We stress the fact that this result only supposes that we have a lower bound for the difference $\widetilde{\alpha}-\widetilde{\beta}$. We think that this is a reasonable hypothesis since it only ensures that the inverse Reynold numbers Re^{-1} and Rm^{-1} are both bounded below. The proof of this result is postponed to appendix A.

Lemma 3.3. *Let* $(\widetilde{\mathbf{v}}_i, \widetilde{\mathbf{w}}_i)$ *be solutions of* (10) *using* $\widetilde{\alpha}_i, \widetilde{\beta}_i > 0$ *and* ϕ_i^v, ϕ_i^w, $i = 1, 2$. *Define* $\overline{\mathbf{v}} = \frac{1}{2}(\widetilde{\mathbf{v}}_1 + \widetilde{\mathbf{v}}_2)$, $\overline{\mathbf{w}} = \frac{1}{2}(\widetilde{\mathbf{w}}_1 + \widetilde{\mathbf{w}}_2)$, $\overline{\alpha} = \frac{1}{2}(\widetilde{\alpha}_1 + \widetilde{\alpha}_2)$, $\overline{\beta} = \frac{1}{2}(\widetilde{\beta}_1 + \widetilde{\beta}_2)$. *For any* $p \in [0, 1]$, *define*

$$\gamma = \overline{\alpha}\left(1 - \frac{|\widetilde{\alpha}_1 - \widetilde{\alpha}_2|^p}{2\overline{\alpha}^p}\right) - \overline{\beta}\left(1 - \frac{|\widetilde{\beta}_1 - \widetilde{\beta}_2|^p}{2\overline{\beta}^p}\right)$$

Fix $\nu_{\min} > 0$ *and suppose that:*

$$\widetilde{\alpha}_i - \widetilde{\beta}_i > \nu_{\min},\ i = 1, 2,\ \mu > \frac{N^2}{c_P^2}\gamma,\ N > \frac{8c_P}{\gamma} M_V(\phi_i^v, \phi_i^w),\ i = 1, 2.$$

Then $\gamma \in [\nu_{\min}, \frac{\overline{\alpha}-\overline{\beta}}{2}]$ *and*

$$\begin{aligned} &\|\widetilde{\mathbf{v}}_1 - \widetilde{\mathbf{v}}_2\|_{C_b(H)}^2 + \|\widetilde{\mathbf{w}}_1 - \widetilde{\mathbf{w}}_2\|_{C_b(H)}^2 \leq \\ &\quad \frac{4c_P^2}{\gamma N^2}\Big(\left(\overline{\alpha}^{1-p}|\widetilde{\alpha}_1 - \widetilde{\alpha}_2|^{2-p} + \overline{\beta}^{1-p}|\widetilde{\beta}_1 - \widetilde{\beta}_2|^{2-p}\right)(\|\overline{\mathbf{w}}\|_{C_b(V)}^2 + \|\overline{\mathbf{v}}\|_{C_b(V)}^2) \\ &\qquad + \frac{\mu^2 c_P^2}{\gamma N^2}(\|\phi_1^v - \phi_2^v\|_{C_b(H)}^2 + \|\phi_1^w - \phi_2^w\|_{C_b(H)}^2)\Big) \end{aligned} \tag{15}$$

If we add an upper bound on $\widetilde{\alpha} - \widetilde{\beta}$, we can get rid on the dependence of μ on γ. Note that $\nu_{\min}$ can be taken close to 0, and $\nu_{\max}$ can be arbitrarily large.

Corollary 3.1. *Let* $(\widetilde{\mathbf{v}}_i, \widetilde{\mathbf{w}}_i)$ *be solutions of* (10) *using* $\widetilde{\alpha}_i, \widetilde{\beta}_i > 0$ *and* ϕ_i^v, ϕ_i^w, $i = 1, 2$. *Define* $\overline{\alpha} = \frac{1}{2}(\widetilde{\alpha}_1 + \widetilde{\alpha}_2)$, $\overline{\beta} = \frac{1}{2}(\widetilde{\beta}_1 + \widetilde{\beta}_2)$. *Fix* $0 < \nu_{\min} < \nu_{\max}$ *and suppose that:*

$$\overline{\alpha} - \overline{\beta} \in (\nu_{\min}, \nu_{\max}),\ \mu > \frac{\nu_{\max} N^2}{c_P^2},\ N > \frac{8c_P}{\nu_{\min}} M_V(\phi_i^v, \phi_i^w),\ i = 1, 2.$$

Then (15) *holds.*

Proof Remark that we have the inequalities $\frac{\nu_{\min}}{2} \leq \gamma \leq \overline{\alpha} - \overline{\beta} \leq \nu_{\max}$. Thus $\mu > \frac{N^2}{c_P^2}\nu_{\max} \geq \frac{N^2}{c_P^2}\gamma$ and $N > \frac{8c_P}{\nu_{\min}} M_V(\phi_i^v, \phi_i^w) \geq \frac{8c_P}{\gamma} M_V(\phi_i^v, \phi_i^w)$.

We may now show the well-posedness of the determining map.

Theorem 3.1. *Let* $R > 0$ *and* $0 < \nu_{\min} < \nu_{\max}$. *Let* $f, g \in C_b(V)$. *Define the set* $\mathcal{D} = \{(\alpha, \beta) \in (\mathbb{R}_{>0})^2 | \nu_{\min} \leq \alpha - \beta \leq \nu_{\max}\}$. *If*

$$N > \frac{8c_P}{\nu_{\min}} \exp\left(\frac{27(c_5^B)^4}{32\nu_{\min}^5}(M_H^R)^4\right)\left(\frac{8}{c_P^2}(M_H^R)^2 + \frac{c_P^4}{2\nu_{\min}^2}(\|f\|_{C_B(V)}^2 + \|g\|_{C_B(V)}^2) + \frac{\mu^2 c_P^4}{\nu_{\min}^2} R^2\right)$$

where $(M_H^R)^2 = \left(\frac{2c_P^2}{\nu_{\min}}\right)^2 \left(\|f\|_{C_B(V)}^2 + \|g\|_{C_B(V)}^2 + \mu^2 R^2\right)$, *and* $\mu \geq \frac{\nu_{\max} N^2}{c_P^2}$, *then* $\mathbb{W} : \mathcal{D} \times (B_R(0))^2 \to (C_b(V))^2$ *is well-defined.*

Proof The proof of existence of solutions on the attractor for any given tuple $(\widetilde{\alpha}, \widetilde{\beta}, \phi^v, \phi^w)$ on the domain of $\mathbb{W}$ is very similar to the case of classical 2D MHD, as it is done in [29]. We therefore omit it.

We prove the uniqueness of solutions of (10) in $C_b(V)$. Given $(\widetilde{\alpha}, \widetilde{\beta}, \phi^v, \phi^w) \in \mathcal{D} \times (B_R(0))^2$, consider $\widetilde{\mathbf{v}}_1$ and $\widetilde{\mathbf{v}}_2$ two solutions of (10) using data $(\widetilde{\alpha}, \widetilde{\beta}, \phi^v, \phi^w) \in \mathcal{D} \times (B_R(0))^2$. We respect the conditions to apply corollary 3.1. Thus, it implies that:

$$\|\widetilde{\mathbf{v}}_1 - \widetilde{\mathbf{v}}_2\|_{C_b(H)} = \|\widetilde{\mathbf{w}}_1 - \widetilde{\mathbf{w}}_2\|_{C_b(H)} = 0.$$

and thus, by the continuous embedding $V \subset H$:

$$\|\widetilde{\mathbf{v}}_1 - \widetilde{\mathbf{v}}_2\|_{C_b(V)} = \|\widetilde{\mathbf{w}}_1 - \widetilde{\mathbf{w}}_2\|_{C_b(V)} = 0.$$

Remark 3.1. *The assumption $\widetilde{\alpha} - \widetilde{\beta} > \nu_{\min}$ means that $\min\{\widehat{Re}^{-1}, \widehat{Rm}^{-1}\} > \nu_{\min}$, where $\widehat{Re}$ (resp. $\widehat{Rm}$) is an approximate value of Re (resp. Rm). Note that this also implies that $\max\{\widehat{Re}^{-1}, \widehat{Rm}^{-1}\} > \nu_{\min}$. Also, in the case of an upper bound, we have that $\min\{\widehat{Re}^{-1}, \widehat{Rm}^{-1}\} < \nu_{\max}$. However, it gives no upper bound on $\max\{\widehat{Re}^{-1}, \widehat{Rm}^{-1}\}$. This will be the case when we will bound $\widetilde{\alpha} + \widetilde{\beta}$.*

4. Parameter recovery inverse problem

Now that $\mathbb{W}$ has been proved to be lipschitz continuous, we focus on the inverse problem of identifying the parameters α and β of some observations $P_N\mathbf{v}$ and $P_N\mathbf{w}$. Most of these results will be stated using general observations $\phi^v, \phi^w \in C_b(V)$, and will then be specialized with filtered solutions of (4).

4.1. Problem definition and existence of solution

First of all, we express the inverse problem as the optimization problem (16), and easily prove that it admits a solution.

Theorem 4.1. *Let $0 < \nu_{\min} < \nu_{\max}$ and $0 < \nu'_{\min} < \nu'_{\max}$. Define the set $\mathcal{D}' = \{(\alpha, \beta) \in (\mathbb{R}_{>0})^2 | \nu_{\min} \le \alpha - \beta \le \nu_{\max}$ and $\nu'_{\min} \le \alpha + \beta \le \nu'_{\max}\}$. For some $R > 0$, let $\phi^v, \phi^w \in B_R(0)$. Suppose N and μ satisfy the conditions of theorem 3.1. The minimization problem*

$$\min_{(\alpha,\beta)\in\mathcal{D}'} \mathcal{J}(\alpha, \beta) = \left\| P_N \mathbb{W}(\alpha, \beta, \phi^v, \phi^w) - \begin{pmatrix} \phi^v \\ \phi^w \end{pmatrix} \right\|_{C_b(H)} \tag{16}$$

admits a solution.

Proof As proved in corollary 3.1, $\mathbb{W}(\cdot, \cdot, \phi^v, \phi^w)$ is a Lipschitz continuous operator. Thus, it proves that $\mathcal{J} : \mathcal{D}' \to \mathbb{R}$ is a continuous operator. Since $\mathcal{D}'$ is compact, by the extreme value theorem, $\mathcal{J}$ admits a minimum on $\mathcal{D}'$.

As stated in [5, Fact 4.2], the use of P_N makes the whole problem interesting. Indeed, suppose $(\mathbf{v}, \mathbf{w})$ is the solution of (4) using parameters (α^*, β^*). If one tries to solve:

$$\min_{(\alpha,\beta)\in\mathcal{D}'} \left\| \mathbb{W}(\alpha, \beta, \phi^v, \phi^w) - \begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} \right\|_{C_b(H)}$$

then one easily proves that the unique solution is (α^*, β^*) ; however, it needs to know entirely $\mathbf{v}$ and $\mathbf{w}$. In (16), we instead choose to see under which minimal amount of data on $\mathbf{v}$ and $\mathbf{w}$ the problem is well-posed.

4.2. Uniqueness of solutions

In order to prove the well-posedness of (16), we start by proving the uniqueness of solution. We start by providing an inequality involving the sum and difference of the observations. The proof of this result is postponed to appendix A.

Lemma 4.1. *Let $f, g, \phi^v, \phi^w \in C_b(V)$ and $\alpha_i, \beta_i \in \mathcal{D}'$, $i = 1, 2$. Fix μ and N as in corollary 3.1. If*

$$\|P_N \mathbb{W}(\alpha_1, \beta_1, \phi^v, \phi^w) - P_N \mathbb{W}(\alpha_2, \beta_2, \phi^v, \phi^w)\|_{C_b(H)} = 0,$$

then, for all $p \in [0, 1]$:

$$|(\widetilde{\alpha}_1 - \widetilde{\alpha}_2) + (\widetilde{\beta}_1 - \widetilde{\beta}_2)||\psi^v + \psi^w| \leq \frac{8c_P^2}{\sqrt{\nu_{\min}}N}(\nu'_{\max})^{\frac{1}{2}-\frac{p}{2}} M_H(\phi^v, \phi^w) M_V(\phi^v, \phi^w) \left(|\widetilde{\alpha}_1 - \widetilde{\alpha}_2|^{2-p} + |\widetilde{\beta}_1 - \widetilde{\beta}_2|^{2-p}\right)^{\frac{1}{2}},$$

$$\begin{aligned} |(\widetilde{\alpha}_1 - &\widetilde{\alpha}_2) - (\widetilde{\beta}_1 - \widetilde{\beta}_2)||\psi^v - \psi^w| \\ &\leq \frac{8c_P^2}{\sqrt{\nu_{\min}}N}(\nu'_{\max})^{\frac{1}{2}-\frac{p}{2}}(M_H(\phi^v, \phi^w) + 2M_V(\phi^v, \phi^w))M_V(\phi^v, \phi^w)\left(|\widetilde{\alpha}_1 - \widetilde{\alpha}_2|^{2-p} + |\widetilde{\beta}_1 - \widetilde{\beta}_2|^{2-p}\right)^{\frac{1}{2}}. \end{aligned}$$

where $(\psi^v, \psi^w) = P_N \mathbb{W}(\alpha_1, \alpha_2, \phi^v, \phi^w) = P_N \mathbb{W}(\alpha_2, \beta_2, \phi^v, \phi^w)$.

Note that in lemma 4.1, if one chooses $\phi^v = P_N \mathbf{v}$ and $\phi^w = P_N \mathbf{w}$, then $\phi^v + \phi^w = 2P_N \mathbf{u}$ and $\phi^v - \phi^w = 2P_N \mathbf{b}$ where $(\mathbf{u}, \mathbf{b})$ is the solution of (3). Thus, we see that in order to draw a conclusion from lemma 4.1, we must make sure that the observed modes of $2\mathbf{u} = \mathbf{v} + \mathbf{w}$ and $2\mathbf{b} = \mathbf{v} - \mathbf{w}$ are not zero. The assumption to have $\mathbf{u} \neq 0$ and $\mathbf{b} \neq 0$ seems natural in order to identify Re^{-1} and Rm^{-1}, since if they were both zero, then the influence of the Reynolds numbers on the solution would be invisible. For a proper proof of why this non vanishing assumption is necessary, see [5, Example 5.4] in the case of the Navier-Stokes equation. In this regard, given $0 \neq \phi \in C_b(H)$ we define the index $n_0(\phi) = \sup\{n \in \mathbb{N} \mid \|P_n(\phi)\|_{C_b(H)} = 0\}$.

Theorem 4.2. *Let $(\mathbf{v}, \mathbf{w}) \in (C_b(V))^2$ be a solution of (4) using parameter $(\alpha^*, \beta^*) \in \mathcal{D}'$. Suppose that $\mathbf{v} + \mathbf{w} \neq 0$ and $\mathbf{v} - \mathbf{w} \neq 0$. Choose $\mu > \frac{\nu_{\max} N^2}{c_P^2}$ and*

$$N > \max\left\{\frac{16c_P}{\nu_{\min}} M_V(\mathbf{v}, \mathbf{w}),\ n_0(\mathbf{v} + \mathbf{w}),\ \frac{C_1(P_N\mathbf{v}, P_N\mathbf{w})}{\|P_N(\mathbf{v} + \mathbf{w})\|_{C_b(H)}}, n_0(\mathbf{v} - \mathbf{w}),\ \frac{C_2(P_N\mathbf{v}, P_N\mathbf{w})}{\|P_N(\mathbf{v} - \mathbf{w})\|_{C_b(H)}}\right\}$$

where $C_1(\mathbf{v}, \mathbf{w}) = \frac{8c_P^2\sqrt{\nu'_{\max}}}{\sqrt{\nu_{\min}}} M_H(\mathbf{v}, \mathbf{w}) M_V(\mathbf{v}, \mathbf{w})$ and $C_2(\mathbf{v}, \mathbf{w}) = \frac{8c_P^2\sqrt{\nu'_{\max}}}{\sqrt{\nu_{\min}}}(M_H(\mathbf{v}, \mathbf{w}) + 2M_V(\mathbf{v}, \mathbf{w}))M_V(\mathbf{v}, \mathbf{w})$. Then (α^*, β^*) is the unique global solution of the minimization problem (16) using $(\phi^v, \phi^w) = (P_N\mathbf{v}, P_N\mathbf{w})$.

Proof Since $(\mathbf{v}, \mathbf{w}) \in (C_b(V))^2$, we can take $R = \max\{\|P_N\mathbf{v}\|_{C_b(V)}, \|P_N\mathbf{w}\|_{C_b(V)}\} < \infty$ and prove using theorem 3.1 to prove that $(\mathbf{v}, \mathbf{w}) = \mathbb{W}(\alpha^*, \beta^*, P_N\mathbf{v}, P_N\mathbf{w})$. Remark also that $\mathcal{J}(\alpha^*, \beta^*) = 0$, thus (α^*, β^*) is a solution of (16). Let $(\widetilde{\alpha}, \widetilde{\beta}) \in \mathcal{D}'$, such that $\mathcal{J}(\widetilde{\alpha}, \widetilde{\beta}) = 0$ and suppose that $(\widetilde{\alpha}, \widetilde{\beta}) \neq (\alpha^*, \beta^*)$. Using lemma 3.3 with $p = 0$, we have the estimates:

$$\begin{aligned} N|(\widetilde{\alpha} - \alpha^*) + (\widetilde{\beta} - \beta^*)||P_N\mathbf{v} + P_N\mathbf{w}| &\leq C_1(P_N\mathbf{v}, P_N\mathbf{w})\left(|\widetilde{\alpha} - \alpha^*|^2 + |\widetilde{\beta} - \beta^*|^2\right)^{\frac{1}{2}} \\ &\leq C_1(P_N\mathbf{v}, P_N\mathbf{w})\left(|\widetilde{\alpha} - \alpha^*| + |\widetilde{\beta} - \beta^*|\right) \end{aligned} \tag{17}$$

$$\begin{aligned} N|(\widetilde{\alpha} - \alpha^*) - (\widetilde{\beta} - \beta^*)||P_N\mathbf{v} - P_N\mathbf{w}| &\leq C_2(P_N\mathbf{v}, P_N\mathbf{w})\left(|\widetilde{\alpha} - \alpha^*|^2 + |\widetilde{\beta} - \beta^*|^2\right)^{\frac{1}{2}} \\ &\leq C_2(P_N\mathbf{v}, P_N\mathbf{w})\left(|\widetilde{\alpha} - \alpha^*| + |\widetilde{\beta} - \beta^*|\right). \end{aligned} \tag{18}$$

We now distinguish 2 cases:

- If $\widetilde{\alpha} - \alpha^* \geq 0$ and $\widetilde{\beta} - \beta^* \geq 0$ (or $\widetilde{\alpha} - \alpha^* \leq 0$ and $\widetilde{\beta} - \beta^* \leq 0$), then from (17):

$$N \leq \frac{C_1(P_N\mathbf{v}, P_N\mathbf{w})}{|P_N\mathbf{v} + P_N\mathbf{w}|} \frac{\widetilde{\alpha} - \alpha^* + \widetilde{\beta} - \beta^*}{(\widetilde{\alpha} - \alpha^*) + (\widetilde{\beta} - \beta^*)} \leq \frac{C_1(P_N\mathbf{v}, P_N\mathbf{w})}{|P_N\mathbf{v} + P_N\mathbf{w}|}.$$

- If $\widetilde{\alpha} - \alpha^* \geq 0$ and $\widetilde{\beta} - \beta^* \leq 0$ (or $\widetilde{\alpha} - \alpha^* \leq 0$ and $\widetilde{\beta} - \beta^* \geq 0$), then from (18):

$$N \leq \frac{C_2(P_N\mathbf{v}, P_N\mathbf{w})}{|P_N\mathbf{v} - P_N\mathbf{w}|} \frac{\widetilde{\alpha} - \alpha^* - (\widetilde{\beta} - \beta^*)}{(\widetilde{\alpha} - \alpha^*) - (\widetilde{\beta} - \beta^*)} \leq \frac{C_2(P_N\mathbf{v}, P_N\mathbf{w})}{|P_N\mathbf{v} - P_N\mathbf{w}|}.$$

In both cases, these inequalities are in contradiction with the hypothesis on N. Thus, $(\widetilde{\alpha}, \widetilde{\beta}) = (\alpha^*, \beta^*)$.

4.3. **Lipschitz continuity of the inverse problem**

We continue the analysis of the well-posedness of the minimization problem (16) by proving a lipschitz property. As we have shown, the map $(P_N\mathbf{v}, P_N\mathbf{w}) \mapsto \arg\min_{\alpha,\beta} \|P_N\mathbb{W}(\alpha, \beta, P_N\mathbf{v}, P_N\mathbf{w}) - (P_N\mathbf{v}, P_N\mathbf{w})\|_{C_B(H)}$ is well defined. We will show in theorem 4.3 that this map is also lipschitz continuous. This property is important in order to make sure that this identification process is stable to perturbations. We start by showing an estimate on the parameters when using any kind of observations ϕ^v, ϕ^w. This result is presented on a time interval $[s,t]$ which is arbitrary. The proof of this result is found in appendix A.

Lemma 4.2. *Let $(\alpha_i, \beta_i) \in \mathcal{D}'$, $i = 1, 2$ and $\phi^v, \phi^w \in C_b(V)$. Let μ and N such that*

$$\mu > \frac{\nu_{\max} N^2}{c_P^2}, \;\; N > \frac{16 c_P}{\nu_{\min}} M_V(\phi^v, \phi^w)$$

Let $(\mathbf{v}_i, \mathbf{w}_i) = \mathbb{W}(\alpha_i, \beta_i, \phi^v, \phi^w)$, $i = 1, 2$. Then for any interval $[s,t] \subset \mathbb{R}$:

$$\begin{aligned} &(|\alpha_1 - \alpha_2| + |\beta_1 - \beta_2|) \left(\inf_{[s,t]} \left||P_N\mathbf{v}_1|^2 - |P_N\mathbf{w}_1|^2\right| - \frac{M}{N} \right) \\ &\leq \widetilde{M} \left(\|P_N\eta\|_{C_b(H)} + \|P_N\zeta\|_{C_b(H)} \right) \end{aligned} \tag{19}$$

where $M = \frac{8\sqrt{2}c_P^2\sqrt{\nu'_{\max}}}{\sqrt{\nu_{\min}}} c_L M_H(\phi^v, \phi^w) M_V(\phi^v, \phi^w)^2$, $\widetilde{M} = 2\|\mathcal{A}^{-1}\partial_t P_N\mathbf{v}_1\|_{C_b(H)} + 2\|\mathcal{A}^{-1}\partial_t P_N\mathbf{w}_1\|_{C_b(H)}$ $+ 2\nu'_{\max}(\|P_N\mathbf{v}_1\|_{C_b(H)} + \|P_N\mathbf{w}_1\|_{C_b(H)}) + 2\mu\frac{1+\delta^{N^2}}{1-\delta^{N^2}}\left(\|\mathcal{A}^{-1}P_N\mathbf{v}_1\|_{C_b(H)} + \|\mathcal{A}^{-1}P_N\mathbf{w}_1\|_{C_b(H)}\right)$, *and* $\delta = e^{-\frac{\mu(t-s)}{N^2}}$

As an immediate consequence of lemma 4.2, we have the

Theorem 4.3. *Let $(\mathbf{v}, \mathbf{w}) \in (C_b(V))^2$ be a solution of (4) using parameter $(\alpha^*, \beta^*) \in \mathcal{D}'$. Suppose that for some $n > 0$ and some time interval $[s,t]$, $\inf_{[s,t]} ||P_n\mathbf{v}|^2 - |P_n\mathbf{w}|^2| \neq 0$. Choose $\mu > \frac{\nu_{\max} N^2}{c_P^2}$ and*

$$N > \max\left\{ \frac{16 c_P}{\nu_{\min}} M_V(\mathbf{v}, \mathbf{w}), \; n, \; 2\frac{M}{\inf_{[s,t]} ||P_n\mathbf{v}|^2 - |P_n\mathbf{w}|^2|} \right\}.$$

Then for any $(\alpha, \beta) \in \mathcal{D}'$,

$$(|\alpha - \alpha^*| + |\beta - \beta^*|) \leq \frac{2\widetilde{M}}{\inf_{[s,t]} ||P_n\mathbf{v}|^2 - |P_n\mathbf{w}|^2|} \mathcal{J}(\alpha, \beta),$$

where in $\mathcal{J}$, we use $(\phi^v, \phi^w) = (P_N\mathbf{v}, P_N\mathbf{w})$, and $M, \widetilde{M}$ are defined in lemma 4.2.

5. Iterative scheme for parameter recovery

Based on the previous results, we now define an iterative scheme in order to recover the parameters α, β from partial observations, and prove its efficiency numerically.

5.1. Scheme definition and proof

Based on the results of [5], we propose the Algorithm 5.1 in order to recover the parameters α^* and β^* from partial observations $(P_N\mathbf{v}^*, P_N\mathbf{w}^*)$. In this algorithm, we denote $\langle\phi\rangle_s^t = \int_s^t e^{-\mu(t-\tau)}\phi(\tau)d\tau$ and, for some $\varepsilon_1 \in (0, \nu_{\min})$, $\varepsilon_1' \in (0, \nu_{\min}')$.

$$\mathcal{D}'_{\varepsilon_1} = \left\{ (\alpha_\varepsilon, \beta_\varepsilon) \in (\mathbb{R}_{>0})^2 \,\middle|\, \begin{matrix} \nu_{\min} - \varepsilon_1 \le \alpha_\varepsilon - \beta_\varepsilon \le \nu_{\max} + \varepsilon_1 \\ \nu'_{\min} - \varepsilon_1 \le \alpha_\varepsilon + \beta_\varepsilon \le \nu'_{\max} + \varepsilon_1 \end{matrix} \right\}.$$

Algorithm 5.1: Parameters recovery from partial observations.

Input : An admissible set $\mathcal{D}'$, a perturbed admissible set $\mathcal{D}'_{\varepsilon_1}$ with $0 < \varepsilon_1 < \nu_{\min}$, $0 < \varepsilon_1' < \nu'_{\min}$.
Input : A time interval $[s, t]$.
Input : N and μ, both sufficiently large.
Input : $(P_N\mathbf{v}^*, P_N\mathbf{w}^*)$ such that $|\langle |P_N\mathbf{v}|^2 - |P_N\mathbf{w}|^2\rangle_s^t| > 0$.
Input : Initial guess $(\alpha, \beta) \in \mathcal{D}'_{\varepsilon_1}$.

repeat
- $(\alpha_0, \beta_0) \leftarrow (\alpha, \beta)$;
- $(\alpha, \beta) \leftarrow (\Gamma^\alpha(\alpha), \Gamma^\beta(\beta))$;
- $\mathsf{Criteria1} \leftarrow \left(|\alpha + \beta - (\alpha_0 + \beta_0)| \le \frac{(\nu'_{\max} - \nu'_{\min})\varepsilon_2'}{\nu'_{\max} - \nu'_{\min} + \varepsilon_1'} \right)$;
- $\mathsf{Criteria2} \leftarrow \left(|\alpha - \beta - (\alpha_0 - \beta_0)| \le \frac{(\nu_{\max} - \nu_{\min})\varepsilon_2}{\nu_{\max} - \nu_{\min} + \varepsilon_1} \right)$;

until Criteria1 *and* Criteria2;
return (α, β);

Function $(\Gamma^\alpha(\alpha), \Gamma^\beta(\beta))$**:**
- $(\mathbf{v}, \mathbf{w}) \leftarrow \mathbb{W}(\alpha, \beta, P_N\mathbf{v}^*, P_N\mathbf{w}^*)$;
- $\rho_\eta^v \leftarrow \langle \mathcal{A}^{-1}P_N(\mathbf{v}^* - \mathbf{v}), P_N\mathbf{v}^*\rangle$;
- $\rho_\zeta^w \leftarrow \langle \mathcal{A}^{-1}P_N(\mathbf{w}^* - \mathbf{w}), P_N\mathbf{w}^*\rangle$;
- $\rho_\eta^w \leftarrow \langle \mathcal{A}^{-1}P_N(\mathbf{v}^* - \mathbf{v}), P_N\mathbf{w}^*\rangle$;
- $\rho_\zeta^v \leftarrow \langle \mathcal{A}^{-1}P_N(\mathbf{w}^* - \mathbf{w}), P_N\mathbf{v}^*\rangle$;
- $c_1^\alpha \leftarrow \rho_\eta^v(t) - \rho_\zeta^w(t) - e^{-\mu(t-s)}(\rho_\eta^v(s) - \rho_\zeta^w(s))$;
- $c_2^\alpha \leftarrow \langle \langle P_N(\mathbf{v}^* - \mathbf{v}), \mathcal{A}^{-1}\partial_t P_N\mathbf{v}^*\rangle - \langle P_N(\mathbf{w}^* - \mathbf{w}), \mathcal{A}^{-1}\partial_t P_N\mathbf{w}^*\rangle\rangle_s^t$;
- $c_3^\alpha \leftarrow$ $-\alpha\langle\langle P_N(\mathbf{v}^* - \mathbf{v}), P_N\mathbf{v}^*\rangle - \langle P_N(\mathbf{w}^* - \mathbf{w}), P_N\mathbf{w}^*\rangle\rangle_s^t - \beta\langle\langle P_N(\mathbf{w}^* - \mathbf{w}), P_N\mathbf{v}^*\rangle - \langle P_N(\mathbf{v}^* - \mathbf{v}), P_N\mathbf{w}^*\rangle\rangle_s^t$;
- $c_1^\beta \leftarrow \rho_\eta^w(t) - \rho_\zeta^v(t) - e^{-\mu(t-s)}(\rho_\eta^w(s) - \rho_\zeta^v(s))$;
- $c_2^\beta \leftarrow -\langle \langle P_N(\mathbf{v}^* - \mathbf{v}), \mathcal{A}^{-1}\partial_t P_N\mathbf{w}^*\rangle + \langle P_N(\mathbf{w}^* - \mathbf{w}), \mathcal{A}^{-1}\partial_t P_N\mathbf{v}^*\rangle\rangle_s^t$;
- $c_3^\beta \leftarrow \alpha\langle\langle P_N(\mathbf{v}^* - \mathbf{v}), P_N\mathbf{w}^*\rangle - \langle P_N(\mathbf{w}^* - \mathbf{w}), P_N\mathbf{v}^*\rangle\rangle_s^t + \beta\langle\langle P_N(\mathbf{w}^* - \mathbf{w}), P_N\mathbf{w}^*\rangle - \langle P_N(\mathbf{v}^* - \mathbf{v}), P_N\mathbf{v}^*\rangle\rangle_s^t$
- **return** $\left(\alpha - \frac{c_1^\alpha + c_2^\alpha + c_3^\alpha}{\langle |P_N\mathbf{v}^*|^2 - |P_N\mathbf{w}^*|^2\rangle_s^t}, \beta - \frac{c_1^\beta + c_2^\beta + c_3^\beta}{\langle |P_N\mathbf{v}^*|^2 - |P_N\mathbf{w}^*|^2\rangle_s^t} \right)$;

The theorem 5.1 proves the convergence of Algorithm 5.1 under some hypothesis on N and μ.

Theorem 5.1. *Let* $(\alpha^*,\beta^*)\in\mathcal{D}'$*, and denote* $(\mathbf{v}^*,\mathbf{w}^*)$ *the solution of* (5) *using* (α^*,β^*)*. Suppose there exists* $n>0$ *and an interval* $[s,t]$ *such that* $|\langle|P_n\mathbf{v}^*|^2-|P_n\mathbf{w}^*|^2\rangle_s^t|>0$*. Let* $\varepsilon_1\in(0,\nu_{\min})$*,* $\varepsilon_1'\in(0,\nu'_{\min})$*. Let* $\mu>\frac{\nu_{\max}N^2}{c_P^2}$ *and* N *such that*

$$N>\max\left\{\frac{16c_P}{\nu_{\min}}M_V(P_n\mathbf{v}^*,P_n\mathbf{w}^*),n,\frac{\max(\nu_{\max}-\nu_{\min}+\varepsilon_1,\nu'_{\max}-\nu'_{\min}+\varepsilon'_1)}{\min(\varepsilon_1,\varepsilon'_1)}\frac{M(t-s)}{|\langle|P_n\mathbf{v}^*|^2-|P_n\mathbf{w}^*|^2\rangle_s^t|}\right\}$$

where $M=\frac{4\sqrt{2}c_P^2\sqrt{\nu'_{\max}}}{\sqrt{\nu_{\min}}}c_LM_H(P_n\mathbf{v}^*,P_n\mathbf{w}^*)M_V(P_n\mathbf{v}^*,P_n\mathbf{w}^*)^2$*. Then, for any choice* $(\alpha,\beta)\in\mathcal{D}'_{\varepsilon_1}$*, with* Γ^α *and* Γ^β *defined in Algorithm 5.1, there exists* $\delta\in(0,1)$ *such that:*

$$\begin{cases}|\alpha^*+\beta^*-(\Gamma^\alpha(\alpha)+\Gamma^\beta(\beta))|\le\delta(|\alpha^*-\alpha|+|\beta^*-\beta|),\\ |\alpha^*-\beta^*-(\Gamma^\alpha(\alpha)-\Gamma^\beta(\beta))|\le\delta(|\alpha^*-\alpha|+|\beta^*-\beta|),\end{cases}$$

and $(\Gamma^\alpha(\alpha),\Gamma^\beta(\beta))\in\mathcal{D}'_{\varepsilon_1}$*. Thus, if one denotes* $(\alpha^{k+1},\beta^{k+1})=(\Gamma^\alpha(\alpha^k),\Gamma^\beta(\beta^k))$ *with* $(\alpha^0,\beta^0)\in\mathcal{D}'_{\varepsilon_1}$*, then* $(\alpha^k,\beta^k)\xrightarrow[k\to+\infty]{}(\alpha^*,\beta^*)$*.*

Furthermore, let $\varepsilon_2>0$ *and* $\varepsilon_2'>0$ *be tolerances for stopping. We can infer closeness to the optimal parameters by examining the residuals:*

$$\begin{cases}|\alpha^{k+1}+\beta^{k+1}-(\alpha^k+\beta^k)|\le\dfrac{(\nu'_{\max}-\nu'_{\min})\varepsilon'_2}{\nu'_{\max}-\nu'_{\min}+\varepsilon'_1},\\ |\alpha^{k+1}-\beta^{k+1}-(\alpha^k-\beta^k)|\le\dfrac{(\nu_{\max}-\nu_{\min})\varepsilon_2}{\nu_{\max}-\nu_{\min}+\varepsilon_1},\end{cases}\Longrightarrow\begin{cases}|\alpha^k+\beta^k-(\alpha^*+\beta^*)|\le\varepsilon_2\\ |\alpha^k-\beta^k-(\alpha^*-\beta^*)|\le\varepsilon'_2\end{cases}$$

Proof Let $(\alpha,\beta)\in\mathcal{D}'_{\varepsilon_1}$ and denote $(\mathbf{v},\mathbf{w})=\mathbb{W}(\alpha,\beta,P_N\mathbf{v}^*,P_N\mathbf{w}^*)$. Let $\eta=\mathbf{v}^*-\mathbf{v}$, $\zeta=\mathbf{w}^*-\mathbf{w}$, $\psi^v=P_N\mathbf{v}^*$ and $\psi^w=P_N\mathbf{w}^*$, $\rho_\eta^v=\langle\mathcal{A}^{-1}P_N\eta,\psi^v\rangle$, $\rho_\zeta^w=\langle\mathcal{A}^{-1}P_N\zeta,\psi^w\rangle$, $\rho_\eta^w=\langle\mathcal{A}^{-1}P_N\eta,\psi^w\rangle$, $\rho_\zeta^v=\langle\mathcal{A}^{-1}P_N\zeta,\psi^v\rangle$. We resume the proof from equation (26), which reads as:

$$\begin{aligned}0=\rho_\eta^v(t)-\rho_\zeta^w(t)&-e^{-\mu(t-s)}(\rho_\eta^v(s)-\rho_\zeta^w(s))\\ &+(\alpha^*-\alpha)\langle|\psi^v|^2-|\psi^w|^2\rangle_s^t\\ &-\langle\langle P_N\eta,\mathcal{A}^{-1}\partial_t\psi^v\rangle-\langle P_N\zeta,\mathcal{A}^{-1}\partial_t\psi^w\rangle\rangle_s^t\\ &+\alpha\langle\langle\eta,\psi^v\rangle-\langle\zeta,\psi^w\rangle\rangle_s^t+\beta\langle\langle\zeta,\psi^v\rangle-\langle\eta,\psi^w\rangle\rangle_s^t\\ &+\langle\langle\mathcal{B}(\mathbf{v}^*,\zeta)+\mathcal{B}(\eta,\mathbf{w}),\mathcal{A}^{-1}\psi^v\rangle-\langle\mathcal{B}(\mathbf{w}^*,\eta)+\mathcal{B}(\zeta,\mathbf{v}),\mathcal{A}^{-1}\psi^w\rangle\rangle_s^t.\end{aligned}$$

After some computations:

$$\alpha^*-\Gamma^\alpha(\alpha)=-\frac{\langle\langle\mathcal{B}(\mathbf{v}^*,\zeta)+\mathcal{B}(\eta,\mathbf{w}),\mathcal{A}^{-1}\psi^v\rangle-\langle\mathcal{B}(\mathbf{w}^*,\eta)+\mathcal{B}(\zeta,\mathbf{v}),\mathcal{A}^{-1}\psi^w\rangle\rangle_s^t}{\langle|\psi^v|^2-|\psi^w|^2\rangle_s^t}.$$

where

$$\begin{aligned}\Gamma^\alpha(\alpha)=&\alpha-\frac{\rho_\eta^v(t)-\rho_\zeta^w(t)-e^{-\mu(t-s)}(\rho_\eta^v(s)-\rho_\zeta^w(s))}{\langle|\psi^v|^2-|\psi^w|^2\rangle_s^t}\\ &+\frac{\langle\langle P_N\eta,\mathcal{A}^{-1}\partial_t\psi^v\rangle-\langle P_N\zeta,\mathcal{A}^{-1}\partial_t\psi^w\rangle\rangle_s^t}{\langle|\psi^v|^2-|\psi^w|^2\rangle_s^t}\\ &-\frac{\alpha\langle\langle\eta,\psi^v\rangle-\langle\zeta,\psi^w\rangle\rangle_s^t+\beta\langle\langle\zeta,\psi^v\rangle-\langle\eta,\psi^w\rangle\rangle_s^t}{\langle|\psi^v|^2-|\psi^w|^2\rangle_s^t}\end{aligned}$$

Similarly, starting from the equation (30):

$$
\begin{aligned}
0 = \rho^w_\eta(t) - \rho^v_\zeta(t) - & e^{-\mu(t-s)}(\rho^w_\eta(s) - \rho^v_\zeta(s)) \\
& +(\beta^* - \beta)\langle|\psi^w|^2 - |\psi^v|^2\rangle^t_s \\
& -\langle\langle P_N\eta, \mathcal{A}^{-1}\partial_t\psi^w\rangle - \langle P_N\zeta, \mathcal{A}^{-1}\partial_t\psi^v\rangle\rangle^t_s \\
& +\alpha\langle\langle\eta,\psi^w\rangle - \langle\zeta,\psi^v\rangle\rangle^t_s + \beta\langle\langle\zeta,\psi^w\rangle - \langle\eta,\psi^v\rangle\rangle^t_s \\
& +\langle\langle\mathcal{B}(\mathbf{v}^*,\zeta) + \mathcal{B}(\eta,\mathbf{w}), \mathcal{A}^{-1}\psi^w\rangle - \langle\mathcal{B}(\mathbf{w}^*,\eta) + \mathcal{B}(\zeta,\mathbf{v}), \mathcal{A}^{-1}\psi^v\rangle\rangle^t_s.
\end{aligned}
$$

and then with the same computations as before:

$$
\beta^* - \Gamma^\beta(\beta) = \frac{\langle\langle\mathcal{B}(\mathbf{v}^*,\zeta) + \mathcal{B}(\eta,\mathbf{w}), \mathcal{A}^{-1}\psi^w\rangle - \langle\mathcal{B}(\mathbf{w}^*,\eta) + \mathcal{B}(\zeta,\mathbf{v}), \mathcal{A}^{-1}\psi^v\rangle\rangle^t_s}{\langle|\psi^v|^2 - |\psi^w|^2\rangle^t_s}
$$

where

$$
\begin{aligned}
\Gamma^\beta(\beta) = & \beta - \frac{\rho^w_\eta(t) - \rho^v_\zeta(t) - e^{-\mu(t-s)}(\rho^w_\eta(s) - \rho^v_\zeta(s))}{\langle|\psi^w|^2 - |\psi^v|^2\rangle^t_s} \\
& + \frac{\langle\langle P_N\eta, \mathcal{A}^{-1}\partial_t\psi^w\rangle - \langle P_N\zeta, \mathcal{A}^{-1}\partial_t\psi^v\rangle\rangle^t_s}{\langle|\psi^w|^2 - |\psi^v|^2\rangle^t_s} \\
& - \frac{\alpha\langle\langle\eta,\psi^w\rangle - \langle\zeta,\psi^v\rangle\rangle^t_s + \beta\langle\langle\zeta,\psi^w\rangle - \langle\eta,\psi^v\rangle\rangle^t_s}{\langle|\psi^w|^2 - |\psi^v|^2\rangle^t_s}
\end{aligned}
$$

From these equations, we deduce, by addition and substraction:

$$
\begin{aligned}
& \alpha^* + \beta^* - (\Gamma^\alpha(\alpha) + \Gamma^\beta(\beta)) = \\
& \qquad \frac{\langle\langle\mathcal{B}(\mathbf{v}^*,\zeta) + \mathcal{B}(\eta,\mathbf{w}) + \mathcal{B}(\mathbf{w}^*,\eta) + \mathcal{B}(\zeta,\mathbf{v}), \mathcal{A}^{-1}(\psi^w - \psi^v)\rangle\rangle^t_s}{\langle|\psi^v|^2 - |\psi^w|^2\rangle^t_s}
\end{aligned}
$$

$$
\begin{aligned}
& \alpha^* - \beta^* - (\Gamma^\alpha(\alpha) - \Gamma^\beta(\beta)) = \\
& \qquad - \frac{\langle\langle\mathcal{B}(\mathbf{v}^*,\zeta) + \mathcal{B}(\eta,\mathbf{w}) - \langle\mathcal{B}(\mathbf{w}^*,\eta) - \mathcal{B}(\zeta,\mathbf{v}), \mathcal{A}^{-1}(\psi^w + \psi^v)\rangle\rangle^t_s}{\langle|\psi^v|^2 - |\psi^w|^2\rangle^t_s}
\end{aligned}
$$

Following the arguments shown in the proof of lemma 4.2, we prove that $\left|\langle\langle\mathcal{B}(\mathbf{v}^*,\zeta) + \mathcal{B}(\eta,\mathbf{w}) + \mathcal{B}(\mathbf{w}^*,\eta) + \mathcal{B}(\zeta,\mathbf{v}), \mathcal{A}^{-1}(\psi^w - \psi^v)\rangle\rangle^t_s\right| \leq \frac{M(t-s)}{N}(|\alpha^* - \alpha| + |\beta^* - \beta|)$ and $\left|\langle\langle\mathcal{B}(\mathbf{v}^*,\zeta) + \mathcal{B}(\eta,\mathbf{w}) - \mathcal{B}(\mathbf{w}^*,\eta) - \mathcal{B}(\zeta,\mathbf{v}), \mathcal{A}^{-1}(\psi^w + \psi^v)\rangle\rangle^t_s\right| \leq \frac{M(t-s)}{N}(|\alpha^* - \alpha| + |\beta^* - \beta|)$. Thus, one has the estimates:

$$
\begin{cases}
|\alpha^* + \beta^* - (\Gamma^\alpha(\alpha) + \Gamma^\beta(\beta))| \leq \dfrac{M(t-s)}{N|\langle|\psi^v|^2 - |\psi^w|^2\rangle^t_s|}(|\alpha^* - \alpha| + |\beta^* - \beta|) \\
|\alpha^* - \beta^* - (\Gamma^\alpha(\alpha) - \Gamma^\beta(\beta))| \leq \dfrac{M(t-s)}{N|\langle|\psi^v|^2 - |\psi^w|^2\rangle^t_s|}(|\alpha^* - \alpha| + |\beta^* - \beta|)
\end{cases}
$$

Differentiating the cases when $\alpha^* - \alpha$ and $\beta^* - \beta$ have the same sign or opposite signs, one proves that $|\alpha^* - \alpha| + |\beta^* - \beta| \leq m_\nu$ where $m_\nu = \max(\nu_{\max} - \nu_{\min} + \varepsilon_1, \nu'_{\max} - \nu'_{\min} + \varepsilon'_1)$. Choose $N > \frac{M(t-s)}{\delta|\langle|\psi^v|^2 - |\psi^w|^2\rangle^t_s|}$ for some $\delta < 1$. Thus:

$$
\begin{cases}
|\alpha^* + \beta^* - (\Gamma^\alpha(\alpha) + \Gamma^\beta(\beta))| \leq \delta(|\alpha^* - \alpha| + |\beta^* - \beta|) \leq \delta m_\nu, \\
|\alpha^* - \beta^* - (\Gamma^\alpha(\alpha) - \Gamma^\beta(\beta))| \leq \delta(|\alpha^* - \alpha| + |\beta^* - \beta|) \leq \delta m_\nu,
\end{cases}
$$

and using $(\alpha^*, \beta^*) \in \mathcal{D}'$:

$$
\begin{cases}
\nu'_{\min} - \delta m_\nu \leq \Gamma^\alpha(\alpha) + \Gamma^\beta(\beta) & \leq \nu'_{\max} + \delta m_\nu, \\
\nu_{\min} - \delta m_\nu \leq \Gamma^\alpha(\alpha) - \Gamma^\beta(\beta) & \leq \nu_{\max} + \delta m_\nu.
\end{cases}
$$

Choose $\delta = \frac{\min(\varepsilon_1, \varepsilon_1')}{m_\nu}$. Thus:

$$\begin{cases} \nu'_{\min} - \varepsilon'_1 \leq \Gamma^\alpha(\alpha) + \Gamma^\beta(\beta) & \leq \nu'_{\max} + \varepsilon'_1, \\ \nu_{\min} - \varepsilon_1 \leq \Gamma^\alpha(\alpha) - \Gamma^\beta(\beta) & \leq \nu_{\max} + \varepsilon_1. \end{cases}$$

Furthermore, we also verify that $\delta < 1$, since:

$$\delta \leq \frac{\varepsilon_1}{\nu_{\max} - (\nu_{\min} - \varepsilon_1)} \leq \frac{\nu_{\min}}{\nu_{\max}} < 1.$$

Concerning the convergence of $\{(\alpha^k, \beta^k\}$ we prove that:

$$\begin{cases} |\alpha^* + \beta^* - (\alpha^k + \beta^k)| \leq \delta^k m_\nu \xrightarrow[k\to\infty]{} 0, \\ |\alpha^* - \beta^* - (\alpha^k - \beta^k)| \leq \delta^k m_\nu \xrightarrow[k\to\infty]{} 0. \end{cases}$$

Concerning now the closeness to the optimal solution, denote $\omega^+ = \alpha^* + \beta^*$, $\omega^- = \alpha^* - \beta^*$, $\Gamma^{+,k} = \alpha^k + \beta^k$, $\Gamma^{-,k} = \alpha^k - \beta^k$.

$$\begin{cases} |\Gamma^{+,k} - \omega^+| \leq |\Gamma^{+,k+1} - \Gamma^{+,k}| + |\Gamma^{+,k+1} - \omega^+|, \\ |\Gamma^{-,k} - \omega^-| \leq |\Gamma^{-,k+1} - \Gamma^{-,k}| + |\Gamma^{-,k+1} - \omega^-|, \end{cases}$$

so:

$$\begin{cases} |\Gamma^{+,k+1} - \Gamma^{+,k}| \geq & |\Gamma^{+,k} - \omega^+| + |\Gamma^{+,k+1} - \omega^+| \geq & (1-\delta)|\Gamma^{+,k} - \omega^+|, \\ |\Gamma^{-,k-1} - \Gamma^{-,k}| \geq & |\Gamma^{-,k} - \omega^-| + |\Gamma^{-,k+1} - \omega^-| \geq & (1-\delta)|\Gamma^{-,k} - \omega^-|, \end{cases}$$

If after k iterations, $|\Gamma^{+,k} - \omega^+| > \varepsilon_2$ or $|\Gamma^{-,k} - \omega^-| > \varepsilon'_2$, then:

$$\begin{cases} |\Gamma^{+,k+1} - \Gamma^{+,k}| \geq & (1-\delta)\varepsilon'_2 \geq \dfrac{(\nu'_{\max} - \nu'_{\min})\varepsilon'_2}{\nu'_{\max} - \nu'_{\min} + \varepsilon'_1}, \\ & \text{or} \\ |\Gamma^{-,k-1} - \Gamma^{-,k}| \geq & (1-\delta)\varepsilon_2 \geq \dfrac{(\nu_{\max} - \nu_{\min})\varepsilon_2}{\nu_{\max} - \nu_{\min} + \varepsilon_1}, \end{cases}$$

The contrapositive of this implication ends the proof.

5.2. Numerical test

We now test numerically the algorithm 5.1. More exactly, we rewrite the algorithm in terms of the variables $(\mathbf{u}, \mathbf{b})$ and test the hypothesis on N and μ.

5.2.1. *Algorithm in term of velocity and magnetic field*

The analysis done above can be carried with the equations (3) in term of velocity and magnetic field in order to identify Re^{-1} and Rm^{-1} and prove its convergence. We rewrite the algorithm in these terms in Algorithm 5.2. In there, the operator $\mathbb{W}(\mathrm{Re}^{-1}, \mathrm{Rm}^{-1}, \phi^u, \phi^b)$ should be understood as the solutions $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{b}})$ of the equations:

$$\begin{aligned} &\partial_t \widetilde{\mathbf{u}} - \mathrm{Re}^{-1}\Delta\widetilde{\mathbf{u}} + (\widetilde{\mathbf{u}} \cdot \nabla)\widetilde{\mathbf{u}} - (\widetilde{\mathbf{b}} \cdot \nabla)\widetilde{\mathbf{b}} + \nabla p = \tilde{f} + \mu(\phi^u - P_N(\widetilde{\mathbf{u}})) \text{ in } \Omega, \\ &\partial_t \widetilde{\mathbf{b}} - \mathrm{Rm}^{-1}\Delta\widetilde{\mathbf{b}} + (\widetilde{\mathbf{u}} \cdot \nabla)\widetilde{\mathbf{b}} - (\widetilde{\mathbf{b}} \cdot \nabla)\widetilde{\mathbf{u}} = \tilde{g} + \mu(\phi^b - P_N(\widetilde{\mathbf{b}})) \text{ in } \Omega, \\ &\nabla \cdot \widetilde{\mathbf{u}} = \nabla \cdot \widetilde{\mathbf{b}} = 0, \\ &\widetilde{\mathbf{u}}(0),\ \widetilde{\mathbf{b}}(0) \text{ given.} \end{aligned} \tag{20}$$

Algorithm 5.2: Parameters recovery from partial velocity and magnetic field observations.

Input : An admissible set $\mathcal{D}'$, a perturbed admissible set $\mathcal{D}'_{\varepsilon_1}$ with $0 < \varepsilon_1 < \nu_{\min}$, $0 < \varepsilon'_1 < \nu'_{\min}$.
Input : A time interval $[s, t]$.
Input : N and μ, both sufficiently large.
Input : $(P_N \mathbf{u}^*, P_N \mathbf{b}^*)$ such that $\langle |P_N \mathbf{u}^*|^2 \rangle_s^t \neq 0$ and $\langle |P_N \mathbf{b}^*|^2 \rangle_s^t \neq 0$.
Input : Initial guess $(\mathrm{Re}^{-1}, \mathrm{Rm}^{-1}) \in \mathcal{D}'_{\varepsilon_1}$.

repeat
  $(\mathrm{Re}_0^{-1}, \mathrm{Rm}_0^{-1}) \leftarrow (\mathrm{Re}^{-1}, \mathrm{Rm}^{-1})$;
  $(\mathrm{Re}^{-1}, \mathrm{Rm}^{-1}) \leftarrow \Gamma(\mathrm{Re}^{-1}, \mathrm{Rm}^{-1})$;
  $\mathsf{Criteria1} \leftarrow \left(|\mathrm{Re}^{-1} - \mathrm{Re}_0^{-1}| \leq \frac{(\nu'_{\max} - \nu'_{\min})\varepsilon'_2}{\nu'_{\max} - \nu'_{\min} + \varepsilon'_1} \right)$;
  $\mathsf{Criteria2} \leftarrow \left(|\mathrm{Rm}^{-1} - \mathrm{Rm}_0^{-1}| \leq \frac{(\nu_{\max} - \nu_{\min})\varepsilon_2}{\nu_{\max} - \nu_{\min} + \varepsilon_1} \right)$;
until Criteria1 *and* Criteria2;
return (Re^{-1}, Rm^{-1});

Function $\Gamma(Re^{-1}, Rm^{-1})$**:**
  $(\mathbf{u}, \mathbf{b}) \leftarrow \mathbb{W}(\mathrm{Re}^{-1}, \mathrm{Rm}^{-1}, P_N \mathbf{u}^*, P_N \mathbf{b}^*)$;
  $\rho_\eta^u \leftarrow \langle \mathcal{A}^{-1} P_N(\mathbf{u}^* - \mathbf{u}), P_N \mathbf{u}^* \rangle$;
  $\rho_\zeta^b \leftarrow \langle \mathcal{A}^{-1} P_N(\mathbf{b}^* - \mathbf{b}), P_N \mathbf{b}^* \rangle$;
  $c_1^{\mathrm{Re}} \leftarrow \rho_\eta^u(t) - e^{-\mu(t-s)} \rho_\eta^u(s)$;
  $c_2^{\mathrm{Re}} \leftarrow \langle \langle \mathcal{A}^{-1} P_N(\mathbf{u}^* - \mathbf{u}), \partial_t P_N \mathbf{u}^* \rangle \rangle_s^t$;
  $c_3^{\mathrm{Re}} \leftarrow -\mathrm{Re}^{-1} \langle \langle P_N(\mathbf{u}^* - \mathbf{u}), P_N \mathbf{u}^* \rangle \rangle_s^t$;
  $c_1^{\mathrm{Rm}} \leftarrow \rho_\zeta^b(t) - e^{-\mu(t-s)} \rho_\zeta^b(s)$;
  $c_2^{\mathrm{Rm}} \leftarrow \langle \langle \mathcal{A}^{-1} P_N(\mathbf{b}^* - \mathbf{b}), \partial_t P_N \mathbf{b}^* \rangle \rangle_s^t$;
  $c_3^{\mathrm{Rm}} \leftarrow -\mathrm{Rm}^{-1} \langle \langle P_N(\mathbf{b}^* - \mathbf{b}), P_N \mathbf{b}^* \rangle \rangle_s^t$
  return $\left(Re^{-1} - \frac{c_1^{Re} + c_2^{Re} + c_3^{Re}}{\langle |P_N \mathbf{u}^*|^2 \rangle_s^t}, \;\; Rm^{-1} - \frac{c_1^{Rm} + c_2^{Rm} + c_3^{Rm}}{\langle |P_N \mathbf{b}^*|^2 \rangle \rangle_s^t} \right)$;

We will test a parameter recovering scheme in the same framework of [19] where the authors test a data assimilation technique with the 2D MHD equations.

5.2.2. *Reference solution*

We compute a reference solution from which we will recover the viscosity and magnetic resistivity. The trajectory should exhibit some non-trivial, time-dependant trajectory in order to test adequately the data assimilation algorithm. We choose a two-mode force for each equation $f, g : \Omega \to \mathbb{R}^2$, defined by

$$f(x, y) = 2\mathcal{R} \left(\begin{pmatrix} 2 + 2\mathrm{i} \\ 1 + \mathrm{i} \end{pmatrix} e^{\mathrm{i}(x - 2y)} + \begin{pmatrix} -6 \\ 0 \end{pmatrix} e^{3\mathrm{i}y} \right) / M_f,$$
$$g(x, y) = 2\mathcal{R} \left(\begin{pmatrix} 4 - 3\mathrm{i} \\ -\frac{2}{3}(4 - 3\mathrm{i}) \end{pmatrix} e^{\mathrm{i}(2x + 3y)} + \begin{pmatrix} -3 + 7\mathrm{i} \\ \frac{1}{5}(-3 + 7\mathrm{i}) \end{pmatrix} e^{\mathrm{i}(x - 5y)} \right) / M_g,$$

where M_f, M_g are chosen such that $|f| = |g| = 10$. Concerning the Reynolds numbers, we chose $\mathrm{Re}^{-1} = \mathrm{Rm}^{-1} = 0.01$. These values are chosen in order to make a compromise on time-step and the resolution of the spectral grid that we have to take in order to solve the equations.

All of our computations were performed using `dedalus` [9], an open source pseudo-spectral package. An implicit/explicit Runga Kutta 222 time stepping scheme was used ; the linear terms were solved implicitly and the nonlinear terms explicitly. The spectral grid uses 256^2 Fourier modes, and the time-step is set to $dt = 1e-4$.

The equation (3) are solved on the time interval $[0, t_0]$, $t_0 = 729.92$, using $(\mathbf{u}(0), \mathbf{b}(0)) = 0$. Some results in this framework are shown in [19, Section 3], where the authors argue that doubling the number of Fourier modes does not change a lot the computed solution. Afterwards, the equations (3) and (20) are solved on $[s, t] = [t_0, t_0 + 1]$ using $(\widetilde{\mathbf{u}}(0), \widetilde{\mathbf{b}}(0)) = 0$ using $\mu = 100$ and we apply the Algorithm 5.2.

5.2.3. *Numerical results*

Numerically, we will mainly test the assumptions on N found in theorem 5.1. Also, following some remarks in [10], we test if we see a difference if the initial guess on the unknown parameters is above or below the correct values. All results are found in Figures 1 and 2.

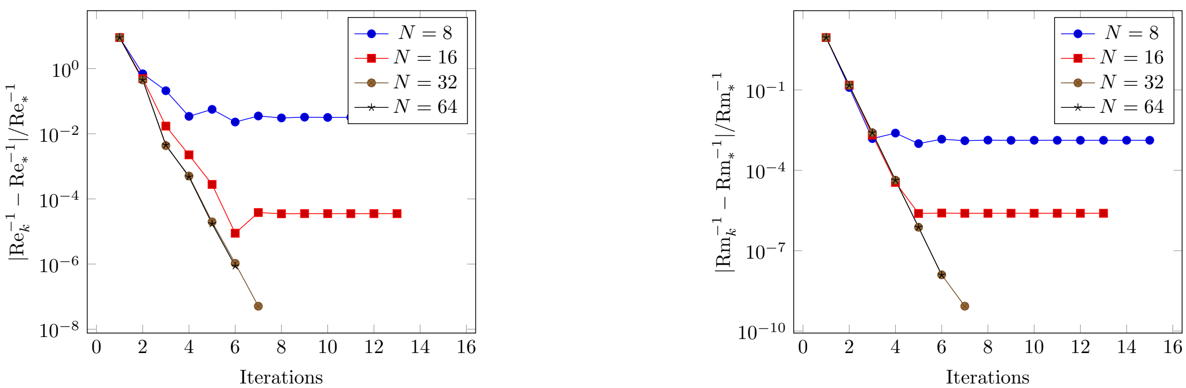

FIGURE 1. Errors on the parameters along iterations using $(\mathrm{Re}_0^{-1}, \mathrm{Rm}_0^{-1}) = (10^{-1}, 10^{-1})$, $(\mathrm{Re}_*^{-1}, \mathrm{Rm}_*^{-1}) = (10^{-2}, 10^{-2})$ and different values of N.

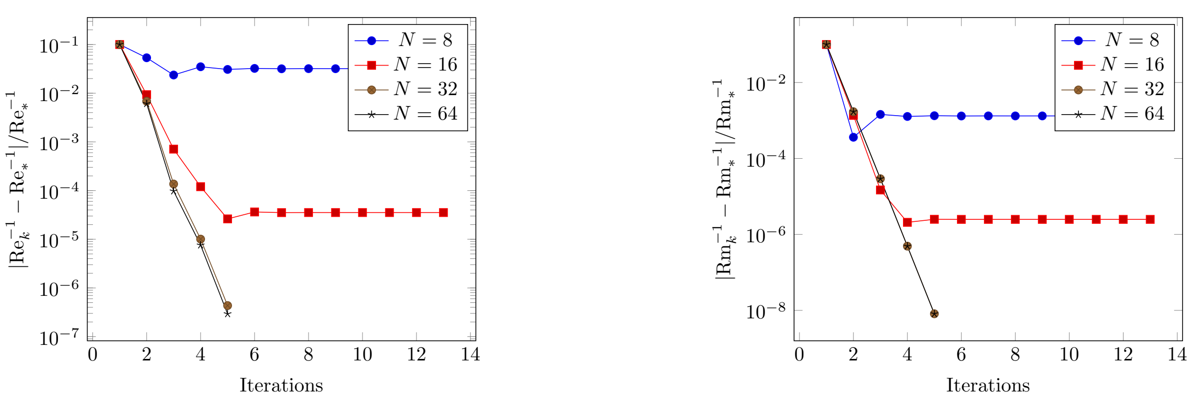

FIGURE 2. Errors on the parameters along iterations using $(\mathrm{Re}_0^{-1}, \mathrm{Rm}_0^{-1}) = (9.10^{-3}, 9.10^{-3})$, $(\mathrm{Re}_*^{-1}, \mathrm{Rm}_*^{-1}) = (10^{-2}, 10^{-2})$ and different values of N.

First of all, we note that, contrary to the observation made in [10], the fact that the initial guess is below or above the true value does not change the behavior of the algorithm. As it can be seen from the figures 1 and 2, all values of N induce a correction of the initial guess. However, for $N \in \{8, 16\}$, the error reaches a plateau preventing any improvement and thus the convergence. The cases $N \in \{32, 64\}$ are also interesting, as they show that doubling (in each direction) the number of nodes does not change the accuracy nor the speed of convergence of the method. We note that in both cases, the convergence is extremely fast.

6. Conclusion

This paper focused on the recovery of the viscosity and magnetic resistivity in the 2D MHD equations based on partial observations. The continuous data assimilation approach reveals to be well suited for this problem, and we proved numerous results which let us build an efficient algorithm. However, all these results rely on the use of Fourier modes observations. The extension of these results to other types of observations and interpolant operators should be a topic of research, given the practical applications it may reach.

Appendix A. Proofs of the previous lemmata

Proof of lemma 3.1 Let $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}})$ be solutions of (10). Taking the inner product of the first line in (10) with $\widetilde{\mathbf{v}}$ gives, using Young's inequality:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\widetilde{\mathbf{v}}|^2 + \widetilde{\alpha}\|\widetilde{\mathbf{v}}\|^2 =& -\widetilde{\beta}\langle \mathcal{A}\widetilde{\mathbf{w}}, \widetilde{\mathbf{v}}\rangle + \langle f, \widetilde{\mathbf{v}}\rangle + \mu\langle \phi^v, \widetilde{\mathbf{v}}\rangle - \mu|P_N\widetilde{\mathbf{v}}|^2 \\ \leq& \frac{\widetilde{\beta}}{2}(\|\widetilde{\mathbf{v}}\|^2 + \|\widetilde{\mathbf{w}}\|^2) + \frac{c_1}{2}|\widetilde{\mathbf{v}}|^2 + \frac{1}{2c_1}|f|^2 - \mu|P_N\widetilde{\mathbf{v}}|^2 + \frac{\mu c_2}{2}|\widetilde{\mathbf{v}}|^2 + \frac{\mu}{2c_2}|\phi^v|^2,\end{aligned}$$

for some arbitrary constants $c_1, c_2 > 0$. Note that:

$$|\widetilde{\mathbf{v}}|^2 = |P_N\widetilde{\mathbf{v}}|^2 + |\widetilde{\mathbf{v}} - P_N\widetilde{\mathbf{v}}|^2 \leq |P_N\widetilde{\mathbf{v}}|^2 + \frac{c_P^2}{N^2}\|\widetilde{\mathbf{v}}\|^2.$$

Thus, after rearranging the terms, we get:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\widetilde{\mathbf{v}}|^2 + \left(\widetilde{\alpha} - \frac{\widetilde{\beta}}{2} - \frac{c_P^2(c_1+\mu c_2)}{2N^2}\right)\|\widetilde{\mathbf{v}}\|^2 - \frac{\widetilde{\beta}}{2}\|\widetilde{\mathbf{w}}\|^2 + \left(\mu - \frac{c_1+\mu c_2}{2}\right)|P_N\widetilde{\mathbf{v}}|^2 \\ \leq \frac{1}{2c_1}|f|^2 + \frac{\mu}{2c_2}|\phi^v|^2,\end{aligned}$$

Similarly, we prove that:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\widetilde{\mathbf{w}}|^2 + \left(\widetilde{\alpha} - \frac{\widetilde{\beta}}{2} - \frac{c_P^2(c_1+\mu c_2)}{2N^2}\right)\|\widetilde{\mathbf{w}}\|^2 - \frac{\widetilde{\beta}}{2}\|\widetilde{\mathbf{v}}\|^2 + \left(\mu - \frac{c_1+\mu c_2}{2}\right)|P_N\widetilde{\mathbf{w}}|^2 \\ \leq \frac{1}{2c_1}|g|^2 + \frac{\mu}{2c_2}|\phi^w|^2.\end{aligned}$$

Summing the two inequalities, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|\widetilde{\mathbf{v}}|^2 + |\widetilde{\mathbf{w}}|^2) &+ \left(\widetilde{\alpha} - \widetilde{\beta} - \frac{c_P^2(c_1+\mu c_2)}{2N^2}\right)(\|\widetilde{\mathbf{v}}\|^2 + \|\widetilde{\mathbf{w}}\|^2) \\ &+ \left(\mu - \frac{c_1+\mu c_2}{2}\right)(|P_N\widetilde{\mathbf{v}}|^2 + |P_N\widetilde{\mathbf{w}}|^2) \\ &\leq \frac{1}{2c_1}(|f|^2 + |g|^2) + \frac{\mu}{2c_2}(|\phi^v|^2 + |\phi^w|^2).\end{aligned}$$

Denote $r = \frac{\nu_{\min} N^2}{\mu c_P^2}$. For some $\varepsilon, \delta \in (0,1)$, we now choose $c_1 = 2\mu r\delta(1-\varepsilon)$ and $c_2 = 2r(1-\delta)$. Thus, $c_1 + \mu c_2 = 2\mu r(1-\delta\varepsilon)$, and the previous differential inequality becomes:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|\widetilde{\mathbf{v}}|^2+|\widetilde{\mathbf{w}}|^2)+\Big(\widetilde{\alpha}-\widetilde{\beta}-\nu_{\min}(1-\delta\varepsilon)\Big)&(\|\widetilde{\mathbf{v}}\|^2+\|\widetilde{\mathbf{w}}\|^2)\\ &+\mu\left(1-r(1-\delta\varepsilon)\right)(|P_N\widetilde{\mathbf{v}}|^2+|P_N\widetilde{\mathbf{w}}|^2)\\ &\leq \frac{1}{2c_1}(|f|^2+|g|^2)+\frac{\mu}{2c_2}(|\phi^v|^2+|\phi^w|^2).\end{aligned} \tag{21}$$

Observe that, due to (2),

$$\|\widetilde{\mathbf{v}}\|^2 \geq \frac{N^2}{c_P^2}|\widetilde{\mathbf{v}}-P_N\widetilde{\mathbf{v}}|^2 = \frac{N^2}{c_P^2}(|\widetilde{\mathbf{v}}|^2-|P_N\widetilde{\mathbf{v}}|^2).$$

Thus,

$$\nu_{\min}\delta\varepsilon\|\widetilde{\mathbf{v}}\|^2+\mu(1-r(1-\delta\varepsilon))|P_N\widetilde{\mathbf{v}}|^2 \geq \mu r\delta\varepsilon|\widetilde{\mathbf{v}}|^2+\mu(1-r)|P_N\widetilde{\mathbf{v}}|^2.$$

Consequently:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|\widetilde{\mathbf{v}}|^2+|\widetilde{\mathbf{w}}|^2)&+\mu r\delta\epsilon(|\widetilde{\mathbf{v}}|^2+|\widetilde{\mathbf{w}}|^2)\\ &+\Big(\widetilde{\alpha}-\widetilde{\beta}-\nu_{\min}\Big)(\|\widetilde{\mathbf{v}}\|^2+\|\widetilde{\mathbf{w}}\|^2)+\mu\left(1-r\right)(|P_N\widetilde{\mathbf{v}}|^2+|P_N\widetilde{\mathbf{w}}|^2)\\ &\leq \frac{1}{2c_1}(|f|^2+|g|^2)+\frac{\mu}{2c_2}(|\phi^v|^2+|\phi^w|^2).\end{aligned}$$

Note that due to the assumption on μ, $r \leq 1$. Thus, we can drop the two last term on the left hand side of the previous inequality, and we prove:

$$\frac{d}{dt}(|\widetilde{\mathbf{v}}|^2+|\widetilde{\mathbf{w}}|^2)+2\frac{\nu_{\min}N^2}{c_P^2}\delta\epsilon(|\widetilde{\mathbf{v}}|^2+|\widetilde{\mathbf{w}}|^2) \leq \frac{1}{c_1}(|f|^2+|g|^2)+\frac{\mu}{c_2}(|\phi^v|^2+|\phi^w|^2).$$

We now choose $\varepsilon = \frac{2}{3}$, $\delta = \frac{3}{4}$, so that $\delta\varepsilon = \frac{1}{2}$, $c_1 = \frac{1}{4}\frac{\nu_{\min}N^2}{c_P^2}$ and $c_2 = \frac{1}{2}\frac{\nu_{\min}N^2}{c_P^2\mu}$. Thus, one has:

$$\begin{aligned}\frac{d}{dt}(|\widetilde{\mathbf{v}}|^2+|\widetilde{\mathbf{w}}|^2)&+\frac{\nu_{\min}N^2}{c_P^2}(|\widetilde{\mathbf{v}}|^2+|\widetilde{\mathbf{w}}|^2)\\ &\leq \frac{4c_P^2}{\nu_{\min}N^2}(\|f\|^2_{C_b(H)}+\|g\|^2_{C_b(H)})+\frac{2\mu^2c_P^2}{\nu_{\min}N^2}(\|\phi^v\|^2_{C_b(H)}+\|\phi^w\|^2_{C_b(H)}).\end{aligned}$$

Denote

$$\mathfrak{p} = \frac{4c_P^2}{\nu_{\min}N^2}(\|f\|^2_{C_b(H)}+\|g\|^2_{C_b(H)})+\frac{2\mu^2c_P^2}{\nu_{\min}N^2}(\|\phi^v\|^2_{C_b(H)}+\|\phi^w\|^2_{C_b(H)}).$$

By Grönwall's lemma, for all $-\infty < s < t < +\infty$:

$$|\widetilde{\mathbf{v}}(t)|^2+|\widetilde{\mathbf{w}}(t)|^2 \leq (|\widetilde{\mathbf{v}}(s)|^2+|\widetilde{\mathbf{w}}(s)|^2)\exp\left(-\frac{\nu_{\min}N^2}{c_P^2}(t-s)\right)+\frac{c_P^2}{N^2\nu_{\min}}\mathfrak{p}.$$

Since by assumption, $\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}} \in C_b(H)$, then $(|\widetilde{\mathbf{v}}(s)|^2 + |\widetilde{\mathbf{w}}(s)|^2) \exp\left(-\frac{\nu_{\min} N^2}{c_P^2}(t-s)\right) \to 0$ when $s \to -\infty$. Thus, for all $t \in \mathbb{R}$:

$$\begin{aligned}|\widetilde{\mathbf{v}}(t)|^2 + |\widetilde{\mathbf{w}}(t)|^2 \leq & \frac{c_P^2}{N^2 \nu_{\min}} \mathfrak{p}, \\ = & \left(\frac{2c_P^2}{\nu_{\min} N^2}\right)^2 (\|f\|^2_{C_b(H)} + \|g\|^2_{C_b(H)}) \\ & + 2\left(\frac{\mu c_P^2}{\nu_{\min} N^2}\right)^2 (\|\phi^v\|^2_{C_b(H)} + \|\phi^w\|^2_{C_b(H)}).\end{aligned}$$

Concerning now (13), we resume the calculations at (21). Choosing $\varepsilon = \frac{4}{5}$, $\delta = \frac{15}{16}$, one proves that:

$$\begin{aligned}&\frac{d}{dt}(|\widetilde{\mathbf{v}}|^2 + |\widetilde{\mathbf{w}}|^2) + \frac{3}{2}\nu_{\min}(\|\widetilde{\mathbf{v}}\|^2 + \|\widetilde{\mathbf{w}}\|^2) \\ &\qquad \leq \frac{4}{3\mu r}(|f|^2 + |g|^2) + \frac{16\mu}{r}(|\phi^v|^2 + |\phi^w|^2).\end{aligned}$$

We now integrate on each side of the inequality, giving, for all $\tau > 0$ and for all $t \in \mathbb{R}$:

$$\begin{aligned}\frac{3}{2}\nu_{\min}\int_{t-\tau}^{t}(\|\widetilde{\mathbf{v}}(s)\|^2 + \|\widetilde{\mathbf{w}}(s)\|^2)ds \leq & \frac{4}{3\mu r}(\|f\|^2_{C_b(H)} + \|g\|^2_{C_b(H)})\tau \\ & + \frac{16\mu}{r}(\|\phi^v\|^2_{C_b(H)} + \|\phi^w\|^2_{C_b(H)})\tau \\ & + |\widetilde{\mathbf{v}}(t-\tau)|^2 + |\widetilde{\mathbf{w}}(t-\tau)|^2, \\ \leq & \frac{4}{3\mu r}(\|f\|^2_{C_b(H)} + \|g\|^2_{C_b(H)})\tau \\ & + \frac{16\mu}{r}(\|\phi^v\|^2_{C_b(H)} + \|\phi^w\|^2_{C_b(H)})\tau \\ & + (M_H(\phi^v, \phi^w))^2.\end{aligned}$$

Proof of lemma 3.2 Let $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}})$ be solutions of (10). Denote $\mathfrak{L}$ the Lebesgue measure on $\mathbb{R}$, and fix $t \in \mathbb{R}$, $\tau > 0$. From (13), we see that, for any $\rho > 0$:

$$\mathfrak{L}(s \in [t-\tau, t] \mid (\|\widetilde{\mathbf{v}}(s)\|^2 + \|\widetilde{\mathbf{w}}(s)\|^2) \geq \rho\} \leq M_V^{L_2}(\phi^v, \phi^w; \tau)^2 \rho^{-1}.$$

Taking $\rho = 2M_V^{L_2}(\phi^v, \phi^w; \tau)^2 \tau^{-1}$, one finds:

$$\mathfrak{L}(s \in [t-\tau, t] \mid (\|\widetilde{\mathbf{v}}(s)\|^2 + \|\widetilde{\mathbf{w}}(s)\|^2) \geq \rho\} \leq \frac{\tau}{2}.$$

Thus, in every interval of length $\tau > 0$, there exists a time $t_0 \in [t-\tau, t]$ such that:

$$\|\widetilde{\mathbf{v}}(t_0)\|^2 + \|\widetilde{\mathbf{w}}(t_0)\|^2 \leq \frac{2}{\tau} M_V^{L_2}(\phi^v, \phi^w; \tau)^2 \tag{22}$$

Now, we test the equations (10) with $(\mathcal{A}\widetilde{\mathbf{v}},\mathcal{A}\widetilde{\mathbf{w}})$; using similar calculations as in lemma 3.1, we prove that there exist constants $c_1, c_2 > 0$ such that:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\widetilde{\mathbf{v}}\|^2+\|\widetilde{\mathbf{w}}\|^2)+&\left(\widetilde{\alpha}-\widetilde{\beta}-\frac{c_P^2}{N^2}\frac{c_1+\mu c_2}{2}\right)(|\mathcal{A}\widetilde{\mathbf{v}}|^2+|\mathcal{A}\widetilde{\mathbf{w}}|^2)\\&+\left(\mu-\frac{c_1+\mu c_2}{2}\right)(\|P_N\widetilde{\mathbf{v}}\|^2+\|P_N\widetilde{\mathbf{w}}\|^2)\\\le&\frac{1}{2c_1}(\|f\|^2_{C_b(V)}+\|g\|^2_{C_b(V)})+\frac{\mu}{2c_2}(\|\phi^v\|^2_{C_b(V)}+\|\phi^w\|^2_{C_b(V)})\\&-\langle\mathcal{B}(\widetilde{\mathbf{w}},\widetilde{\mathbf{v}}),\mathcal{A}\widetilde{\mathbf{v}}\rangle-\langle\mathcal{B}(\widetilde{\mathbf{v}},\widetilde{\mathbf{w}}),\mathcal{A}\widetilde{\mathbf{w}}\rangle\end{aligned}$$

Using (9) and Young's inequality (11) with $p = 4$ and $q = \frac{4}{3}$, one proves that for all $c_3 > 0$:

$$\begin{aligned}\langle\mathcal{B}(\widetilde{\mathbf{w}},\widetilde{\mathbf{v}}),\mathcal{A}\widetilde{\mathbf{v}}\rangle\le&c_5^B|\widetilde{\mathbf{w}}|^{\frac{1}{2}}\|\overline{\mathbf{w}}\|^{\frac{1}{2}}\|\widetilde{\mathbf{v}}\|^{\frac{1}{2}}|\mathcal{A}\widetilde{\mathbf{v}}|^{\frac{3}{2}}\\\le&c_5^B\left(\frac{1}{4c_3^4}|\widetilde{\mathbf{w}}|^2\|\widetilde{\mathbf{w}}\|^2\|\widetilde{\mathbf{v}}\|^2+\frac{3}{4}c_3^{\frac{4}{3}}|\mathcal{A}\widetilde{\mathbf{v}}|^2\right)\\\le&c_5^B\left(\frac{|\widetilde{\mathbf{w}}|^2}{8c_3^4}(\|\widetilde{\mathbf{w}}\|^4+\|\widetilde{\mathbf{v}}\|^4)+\frac{3}{4}c_3^{\frac{4}{3}}|\mathcal{A}\widetilde{\mathbf{v}}|^2\right)\\\le&c_5^B\left(\frac{|\widetilde{\mathbf{w}}|^2}{8c_3^4}(\|\widetilde{\mathbf{w}}\|^2+\|\widetilde{\mathbf{v}}\|^2)^2+\frac{3}{4}c_3^{\frac{4}{3}}|\mathcal{A}\widetilde{\mathbf{v}}|^2\right).\end{aligned}$$

Similarly:

$$\langle\mathcal{B}(\widetilde{\mathbf{v}},\widetilde{\mathbf{w}}),\mathcal{A}\widetilde{\mathbf{w}}\rangle\le c_5^B\left(\frac{|\widetilde{\mathbf{v}}|^2}{8c_3^4}(\|\widetilde{\mathbf{w}}\|^2+\|\widetilde{\mathbf{v}}\|^2)^2+\frac{3}{4}c_3^{\frac{4}{3}}|\mathcal{A}\widetilde{\mathbf{w}}|^2\right).$$

Thus, one has the differential inequality:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\widetilde{\mathbf{v}}\|^2+\|\widetilde{\mathbf{w}}\|^2)+&\left(\widetilde{\alpha}-\widetilde{\beta}-\frac{c_P^2}{N^2}\frac{c_1+\mu c_2}{2}-\frac{3}{4}c_5^Bc_3^{\frac{4}{3}}\right)(|\mathcal{A}\widetilde{\mathbf{v}}|^2+|\mathcal{A}\widetilde{\mathbf{w}}|^2)\\&+\left(\mu-\frac{c_1+\mu c_2}{2}\right)(\|P_N\widetilde{\mathbf{v}}\|^2+\|P_N\widetilde{\mathbf{w}}\|^2)\\\le&\frac{1}{2c_1}(\|f\|^2_{C_b(V)}+\|g\|^2_{C_b(V)})+\frac{\mu}{2c_2}(\|\phi^v\|^2_{C_b(V)}+\|\phi^w\|^2_{C_b(V)})\\&+\frac{c_5^B}{8c_3^4}M_H(\phi^v,\phi^w)^2(\|\widetilde{\mathbf{v}}\|^2+\|\widetilde{\mathbf{w}}\|^2)^2\end{aligned}$$

We choose $c_3=\left(\frac{2(\widetilde{\alpha}-\widetilde{\beta})}{3c_5^B}\right)^{\frac{3}{4}}$ so that $\frac{3}{4}c_5^Bc_3^{\frac{4}{3}}=\frac{\widetilde{\alpha}-\widetilde{\beta}}{2}$. Denoting $C=\frac{27(c_5^B)^4}{32\nu_{\min}^3}$, one proves that $\frac{c_5^B}{8c_3^4}\le\frac{C}{2}$. Now, denoting $r=\frac{\nu_{\min}N^2}{\mu c_P^2}$, we choose $c_1=\mu r\delta(1-\varepsilon)$, $c_2=r(1-\delta)$. Thus, $c_1+\mu c_2=\mu r(1-\delta\varepsilon)$ and the differential inequality becomes:

$$\begin{aligned}\frac{d}{dt}(\|\widetilde{\mathbf{v}}\|^2+\|\widetilde{\mathbf{w}}\|^2)+&\left(\widetilde{\alpha}-\widetilde{\beta}-\nu_{\min}(1-\delta\varepsilon)\right)(|\mathcal{A}\widetilde{\mathbf{v}}|^2+|\mathcal{A}\widetilde{\mathbf{w}}|^2)\\&+\mu\left(2-r(1-\delta\varepsilon)\right)(\|P_N\widetilde{\mathbf{v}}\|^2+\|P_N\widetilde{\mathbf{w}}\|^2)\\\le&\frac{1}{c_1}(\|f\|^2_{C_b(V)}+\|g\|^2_{C_b(V)})+\frac{\mu}{c_2}(\|\phi^v\|^2_{C_b(V)}+\|\phi^w\|^2_{C_b(V)})\\&+CM_H(\phi^v,\phi^w)^2(\|\widetilde{\mathbf{v}}\|^2+\|\widetilde{\mathbf{w}}\|^2)^2\end{aligned}$$

We now use the inequality $|\mathcal{A}\widetilde{\mathbf{v}}|^2 \geq \frac{N^2}{c_P^2}(\|\widetilde{\mathbf{v}}\|^2 - \|P_N\widetilde{\mathbf{v}}\|^2)$ to prove that:

$$\begin{aligned}\nu_{\min}\delta\varepsilon|\mathcal{A}\widetilde{\mathbf{v}}|^2 + \mu(1-r(1-\delta\varepsilon))\|P_N\widetilde{\mathbf{v}}\|^2 \geq& \mu r\delta\varepsilon\|\widetilde{\mathbf{v}}\|^2 + \mu(2-r(1-\delta\varepsilon)-r\delta\varepsilon)\|P_N\widetilde{\mathbf{v}}\|^2\\ \geq& \mu r\delta\varepsilon\|\widetilde{\mathbf{v}}\|^2 + \mu(2-r)\|P_N\widetilde{\mathbf{v}}\|^2.\end{aligned}$$

This implies the inequality:

$$\begin{aligned}\frac{d}{dt}(\|\widetilde{\mathbf{v}}\|^2 + \|\widetilde{\mathbf{w}}\|^2) + &\mu r\delta\varepsilon(\|\widetilde{\mathbf{v}}\|^2 + \|\widetilde{\mathbf{w}}\|^2) + \left(\widetilde{\alpha} - \widetilde{\beta} - \nu_{\min}\right)(|\mathcal{A}\widetilde{\mathbf{v}}|^2 + |\mathcal{A}\widetilde{\mathbf{w}}|^2)\\ &+ \mu\,(2-r)\,(\|P_N\widetilde{\mathbf{v}}\|^2 + \|P_N\widetilde{\mathbf{w}}\|^2)\\ \leq& \frac{1}{c_1}(\|f\|^2_{C_b(V)} + \|g\|^2_{C_b(V)}) + \frac{\mu}{c_2}(\|\phi^v\|^2_{C_b(V)} + \|\phi^w\|^2_{C_b(V)})\\ &+ \frac{c_5^B}{4c_3^4}M_H(\phi^v,\phi^w)^2(\|\widetilde{\mathbf{v}}\|^2 + \|\widetilde{\mathbf{w}}\|^2)^2\end{aligned}$$

Note that $\mu \geq \frac{\nu_{\min}N^2}{2c_P^2}$ is equivalent to $r \leq 2$. Since we suppose that $\widetilde{\alpha} - \widetilde{\beta} \geq \nu_{\min}$, we can get rid of the two last terms in the left-hand side of the inequality. We choose once again $\varepsilon = \frac{2}{3}$ and $\delta = \frac{3}{4}$, which implies that $\delta\varepsilon = \frac{1}{2}$, $c_1 = \frac{1}{4}\mu r$, $c_2 = \frac{r}{4}$. Thus, we proved that:

$$\begin{aligned}\frac{d}{dt}(\|\widetilde{\mathbf{v}}\|^2 + \|\widetilde{\mathbf{w}}\|^2) + \frac{\mu r}{2}(\|\widetilde{\mathbf{v}}\|^2 + \|\widetilde{\mathbf{w}}\|^2) \leq& \frac{4}{\mu r}(\|f\|^2_{C_b(V)} + \|g\|^2_{C_b(V)})\\ &+ \frac{4\mu}{r}(\|\phi^v\|^2_{C_b(V)} + \|\phi^w\|^2_{C_b(V)})\\ &+ CM_H(\phi^v,\phi^w)^2(\|\widetilde{\mathbf{v}}\|^2 + \|\widetilde{\mathbf{w}}\|^2)^2.\end{aligned}$$

For some $t_0 \in \mathbb{R}$ and any $t \in \mathbb{R}$, denote $\psi(t) = \|\widetilde{\mathbf{v}}(t)\|^2 + \|\widetilde{\mathbf{w}}(t)\|^2$, $\Psi(t;t_0) = CM_H(\phi^v,\phi^w)\int_{t_0}^t \psi(s)ds$, and $e_\Psi(t;t_0) = \exp(-\Psi(t;t_0))$. The differential inequality can now be rewritten:

$$\frac{d}{dt}\psi + \frac{\mu r}{2}\psi \leq C' + CM_H(\phi^v,\phi^w)^2\psi^2,$$

where we denote $C' = \frac{4}{\mu r}(\|f\|^2_{C_b(V)} + \|g\|^2_{C_b(V)}) + \frac{4\mu}{r}(\|\phi^v\|^2_{C_b(V)} + \|\phi^w\|^2_{C_b(V)})$. Remark that

$$\begin{aligned}\frac{d}{dt}(\psi(t)e_\Psi(t;t_0)) =& e_\Psi(t;t_0)\left(\frac{d}{dt}\psi(t) - CM_H(\phi^v,\phi^w)\psi(t)^2\right)\\ \leq& e_\Psi(t;t_0)\left(C' - \frac{\mu r}{2}\psi(t)\right)\\ \leq& C'e_\Psi(t;t_0),\end{aligned}$$

since $\psi(t) \geq 0$ for all t. Thus, integrating each side of the inequality, one proves that, for all $t \geq t_0$:

$$\begin{aligned}\psi(t)e_\Psi(t;t_0) \leq& \psi(t_0) + C'\int_{t_0}^t e_\Psi(s;t_0)ds\\ \leq& \psi(t_0) + C'(t-t_0).\end{aligned}$$

Thus:

$$\psi(t) \leq e_\Psi(t;t_0)^{-1}(\psi(t_0) + C'(t-t_0)).$$

In (22) and (13), choose $\tau = \frac{c_P^2}{8\nu_{\min} N^2}$. Note that with this choice of τ, one has $M_V^{L_2}(\phi^v, \phi^w, \tau)^2 \leq \frac{4}{\nu_{\min}} M_H(\phi^v, \phi^w)^2$. Thanks to (22), there exists $t_0 \in [t-\tau, t]$ such that

$$\psi(t_0) \leq \frac{2\nu_{\min}}{c_P^2} M_V^{L_2}(\phi^v, \phi^w, \tau)^2 \leq \frac{8}{c_P^2} M_H(\phi^v, \phi^w)^2.$$

For such t_0, we also prove, thanks to (13), that:

$$\begin{aligned}
\Psi(t; t_0) =& \frac{C}{\nu_{\min}} M_H(\phi^v, \phi^w)^2 \int_{t_0}^t \psi(s) ds \\
\leq& \frac{C}{\nu_{\min}} M_H(\phi^v, \phi^w)^2 \int_{t-\tau}^t \psi(s) ds \\
\leq& \frac{C}{\nu_{\min}} M_H(\phi^v, \phi^w)^2 M_V^{L_2}(\phi^v, \phi^w, \tau)^2 \\
\leq& \frac{4C}{\nu_{\min}^2} M_H(\phi^v, \phi^w)^4.
\end{aligned}$$

Thus, $e_\Psi(t; t_0)^{-1} \leq \exp\left(\frac{4C}{\nu_{\min}^2} M_H(\phi^v, \phi^w)^4\right)$ Thus, we have proven that, for all $t \in \mathbb{R}$:

$$\psi(t) \leq \exp\left(\frac{4C}{\nu_{\min}^2} M_H(\phi^v, \phi^w)^4\right)\left(\frac{8}{c_P^2} M_H(\phi^v, \phi^w)^2 + C'\tau\right).$$

Proof of lemma 4.1 Denote $(\widetilde{\mathbf{v}}_i, \widetilde{\mathbf{w}}_i) = \mathbb{W}(\alpha_i, \beta_i, \phi^v, \phi^w)$, $i = 1, 2$, and assume that $P_N(\widetilde{\mathbf{v}}_1, \widetilde{\mathbf{w}}_1) = P_N(\widetilde{\mathbf{v}}_2, \widetilde{\mathbf{w}}_2) = (\psi^v, \psi^w)$. Writing (10) in Fourier space, for any $\mathbf{k} \in \mathbb{Z}^2$, $|\mathbf{k}| \leq N$, we have, for $i = 1, 2$:

$$\begin{aligned}
\frac{d}{dt}\widehat{\widetilde{\mathbf{v}}}_i(\mathbf{k}) + \frac{|\mathbf{k}|^2}{c_P^2}(\widetilde{\alpha}_i \widehat{\widetilde{\mathbf{v}}}_i(\mathbf{k}) + \widetilde{\beta}_i \widehat{\widetilde{\mathbf{w}}}_i(\mathbf{k})) + \frac{\mathrm{i}}{c_P}\sum_{\mathbf{h}} (\mathbf{h}\widehat{\widetilde{\mathbf{w}}}_i(\mathbf{h}))\widehat{\widetilde{\mathbf{v}}}_i(\mathbf{k}-\mathbf{h}) =& \widehat{f}(\mathbf{k}) + \mu(\widehat{\phi^v}(\mathbf{k}) - \widehat{\widetilde{\mathbf{v}}}_i(\mathbf{k})), \\
\frac{d}{dt}\widehat{\widetilde{\mathbf{w}}}_i(\mathbf{k}) + \frac{|\mathbf{k}|^2}{c_P^2}(\widetilde{\alpha}_i \widehat{\widetilde{\mathbf{w}}}_i(\mathbf{k}) + \widetilde{\beta}_i \widehat{\widetilde{\mathbf{v}}}_i(\mathbf{k})) + \frac{\mathrm{i}}{c_P}\sum_{\mathbf{h}} (\mathbf{h}\widehat{\widetilde{\mathbf{v}}}_i(\mathbf{h}))\widehat{\widetilde{\mathbf{w}}}_i(\mathbf{k}-\mathbf{h}) =& \widehat{g}(\mathbf{k}) + \mu(\widehat{\phi^w}(\mathbf{k}) - \widehat{\widetilde{\mathbf{w}}}_i(\mathbf{k})).
\end{aligned}$$

Denote $\eta = \widetilde{\mathbf{v}}_1 - \widetilde{\mathbf{v}}_2$, $\zeta = \widetilde{\mathbf{w}}_1 - \widetilde{\mathbf{w}}_2$. We subtract the equations for $i = 1$ with the equations for $i = 2$, and we get, using the assumptions on equality of low modes:

$$\begin{aligned}
\frac{|\mathbf{k}|^2}{c_P^2}((\widetilde{\alpha}_1 - \widetilde{\alpha}_2)\widehat{\psi^v}(\mathbf{k}) + (\widetilde{\beta}_1 - \widetilde{\beta}_2)\widehat{\psi^w}(\mathbf{k})) + \frac{\mathrm{i}}{c_P}\sum_{\mathbf{h}} (\mathbf{h}\widehat{\widetilde{\mathbf{w}}}_2(\mathbf{h}))\widehat{\eta}(\mathbf{k}-\mathbf{h}) + (\mathbf{h}\widehat{\zeta}(\mathbf{h}))\widehat{\widetilde{\mathbf{v}}}_1(\mathbf{k}-\mathbf{h}) =& 0, \\
\frac{|\mathbf{k}|^2}{c_P^2}((\widetilde{\alpha}_1 - \widetilde{\alpha}_2)\widehat{\psi^w}(\mathbf{k}) + (\widetilde{\beta}_1 - \widetilde{\beta}_2)\widehat{\psi^v}(\mathbf{k})) + \frac{\mathrm{i}}{c_P}\sum_{\mathbf{h}} (\mathbf{h}\widehat{\eta}(\mathbf{h}))\widehat{\widetilde{\mathbf{w}}}_2(\mathbf{k}-\mathbf{h}) + (\mathbf{h}\widehat{\widetilde{\mathbf{v}}}_1(\mathbf{h}))\widehat{\zeta}(\mathbf{k}-\mathbf{h}) =& 0,
\end{aligned}$$

We add and subtract these two equations in order to get:

$$\begin{aligned}
&\frac{|\mathbf{k}|^2}{c_P^2}((\widetilde{\alpha}_1 - \widetilde{\alpha}_2) + (\widetilde{\beta}_1 - \widetilde{\beta}_2))(\widehat{\psi^v}(\mathbf{k}) + \widehat{\psi^w}(\mathbf{k})) \\
&\qquad + \frac{\mathrm{i}}{c_P}\sum_{\mathbf{h}} (\mathbf{k}\widehat{\widetilde{\mathbf{w}}}_2(\mathbf{k}-\mathbf{h}))\widehat{\eta}(\mathbf{h}) + (\mathbf{k}\widehat{\zeta}(\mathbf{k}-\mathbf{h}))\widehat{\widetilde{\mathbf{v}}}_1(\mathbf{h}) = 0,
\end{aligned}$$

$$\frac{|\mathbf{k}|^2}{c_P^2}((\widetilde{\alpha}_1-\widetilde{\alpha}_2)-(\widetilde{\beta}_1-\widetilde{\beta}_2))(\widehat{\psi^v}(\mathbf{k})-\widehat{\psi^w}(\mathbf{k}))$$
$$+\frac{\mathrm{i}}{c_P}\sum_{\mathbf{h}}((\mathbf{k}-2\mathbf{h})\widehat{\widetilde{\mathbf{w}}}_2(\mathbf{k}-\mathbf{h}))\widehat{\eta}(\mathbf{h})-((\mathbf{k}-2\mathbf{h})\widehat{\widetilde{\mathbf{v}}}_1(\mathbf{k}-\mathbf{h}))\widehat{\zeta}(\mathbf{h})=0,$$

These imply the inequalities:

$$\frac{|\mathbf{k}|^2}{c_P^2}|(\widetilde{\alpha}_1-\widetilde{\alpha}_2)+(\widetilde{\beta}_1-\widetilde{\beta}_2)||\widehat{\psi^v}(\mathbf{k})+\widehat{\psi^w}(\mathbf{k})|$$
$$\leq\frac{|\mathbf{k}|}{c_P}\sum_{\mathbf{h}}\left|\widehat{\widetilde{\mathbf{w}}}_2(\mathbf{k}-\mathbf{h})\widehat{\eta}(\mathbf{h})\right|+\left|\widehat{\zeta}(\mathbf{k}-\mathbf{h})\widehat{\widetilde{\mathbf{v}}}_1(\mathbf{h})\right|,$$

$$\frac{|\mathbf{k}|^2}{c_P^2}|(\widetilde{\alpha}_1-\widetilde{\alpha}_2)-(\widetilde{\beta}_1-\widetilde{\beta}_2)||\widehat{\psi^v}(\mathbf{k})-\widehat{\psi^w}(\mathbf{k})|$$
$$\leq\frac{|\mathbf{k}|}{c_P}\sum_{\mathbf{h}}\left|\widehat{\widetilde{\mathbf{w}}}_2(\mathbf{k}-\mathbf{h})\widehat{\eta}(\mathbf{h})\right|+\left|\widehat{\widetilde{\mathbf{v}}}_1(\mathbf{k}-\mathbf{h})\widehat{\zeta}(\mathbf{h})\right|$$
$$+\frac{2}{c_P}\sum_{\mathbf{h}}\left|((\mathbf{k}-\mathbf{h})\widehat{\widetilde{\mathbf{w}}}_2(\mathbf{k}-\mathbf{h}))\widehat{\eta}(\mathbf{h})\right|+\left|((\mathbf{k}-\mathbf{h})\widehat{\widetilde{\mathbf{v}}}_1(\mathbf{k}-\mathbf{h}))\widehat{\zeta}(\mathbf{h})\right|.$$

Now we estimate the convolutions. Applying the Cauchy-Schwarz inequality and Parseval's theorem, we obtain:

$$\sum_{\mathbf{h}}\left|\widehat{\widetilde{\mathbf{w}}}_2(\mathbf{k}-\mathbf{h})\widehat{\eta}(\mathbf{h})\right|+\left|\widehat{\zeta}(\mathbf{k}-\mathbf{h})\widehat{\widetilde{\mathbf{v}}}_1(\mathbf{h})\right|\leq|\widetilde{\mathbf{w}}_2||\eta|+|\widetilde{\mathbf{v}}_1||\zeta|,$$
$$\sum_{\mathbf{h}}\left|((\mathbf{k}-\mathbf{h})\widehat{\widetilde{\mathbf{w}}}_2(\mathbf{k}-\mathbf{h}))\widehat{\eta}(\mathbf{h})\right|+\left|((\mathbf{k}-\mathbf{h})\widehat{\widetilde{\mathbf{v}}}_1(\mathbf{k}-\mathbf{h}))\widehat{\zeta}(\mathbf{h})\right|\leq\|\widetilde{\mathbf{w}}_2\||\eta|+\|\widetilde{\mathbf{v}}_1\||\zeta|,$$

Thus, we have proved the inequalities:

$$\frac{|\mathbf{k}|^2}{c_P^2}|(\widetilde{\alpha}_1-\widetilde{\alpha}_2)+(\widetilde{\beta}_1-\widetilde{\beta}_2)||\widehat{\psi^v}(\mathbf{k})+\widehat{\psi^w}(\mathbf{k})|\leq\frac{|\mathbf{k}|}{c_P}(|\widetilde{\mathbf{w}}_2||\eta|+|\widetilde{\mathbf{v}}_1||\zeta|),$$

$$\frac{|\mathbf{k}|^2}{c_P^2}|(\widetilde{\alpha}_1-\widetilde{\alpha}_2)-(\widetilde{\beta}_1-\widetilde{\beta}_2)||\widehat{\psi^v}(\mathbf{k})-\widehat{\psi^w}(\mathbf{k})|\leq\frac{|\mathbf{k}|}{c_P}(|\widetilde{\mathbf{w}}_2||\eta|+|\widetilde{\mathbf{v}}_1||\zeta|)$$
$$+\frac{2}{c_P}(\|\widetilde{\mathbf{w}}_2\||\eta|+\|\widetilde{\mathbf{v}}_1\||\zeta|).$$

We multiply each inequality by $\frac{c_P^2|\widehat{\psi^v}(\mathbf{k})\pm\widehat{\psi^w}(\mathbf{k})|}{|\mathbf{k}|^2}$:

$$|(\widetilde{\alpha}_1-\widetilde{\alpha}_2)+(\widetilde{\beta}_1-\widetilde{\beta}_2)||\widehat{\psi^v}(\mathbf{k})+\widehat{\psi^w}(\mathbf{k})|^2\leq\frac{c_P}{|\mathbf{k}|}(|\widetilde{\mathbf{w}}_2||\eta|+|\widetilde{\mathbf{v}}_1||\zeta|)|\widehat{\psi^v}(\mathbf{k})+\widehat{\psi^w}(\mathbf{k})|,$$

$$|(\widetilde{\alpha}_1-\widetilde{\alpha}_2)-(\widetilde{\beta}_1-\widetilde{\beta}_2)||\widehat{\psi^v}(\mathbf{k})-\widehat{\psi^w}(\mathbf{k})|^2\leq\frac{c_P}{|\mathbf{k}|}(|\widetilde{\mathbf{w}}_2||\eta|+|\widetilde{\mathbf{v}}_1||\zeta|)|\widehat{\psi^v}(\mathbf{k})-\widehat{\psi^w}(\mathbf{k})|$$
$$+\frac{2c_P}{|\mathbf{k}|^2}(\|\widetilde{\mathbf{w}}_2\||\eta|+\|\widetilde{\mathbf{v}}_1\||\zeta|)|\widehat{\psi^v}(\mathbf{k})-\widehat{\psi^w}(\mathbf{k})|.$$

Take now the sum for all $\mathbf{k}\in\mathbb{Z}^2$, $0<|\mathbf{k}|\leq N$, and use Cauchy-Schwartz in order to prove:

$$|(\widetilde{\alpha}_1-\widetilde{\alpha}_2)+(\widetilde{\beta}_1-\widetilde{\beta}_2)||\psi^v+\psi^w|^2\leq c_P(|\widetilde{\mathbf{w}}_2||\eta|+|\widetilde{\mathbf{v}}_1||\zeta|)|\psi^v-\psi^w|\left(\sum_{0<|\mathbf{k}|\leq N}\frac{1}{|\mathbf{k}|^2}\right)^{\frac{1}{2}},$$

$$|(\widetilde{\alpha}_1-\widetilde{\alpha}_2)-(\widetilde{\beta}_1-\widetilde{\beta}_2)||\psi^v-\psi^w|^2 \leq c_P(|\widetilde{\mathbf{w}}_2||\eta|+|\widetilde{\mathbf{v}}_1||\zeta|)|\psi^v-\psi^w|\left(\sum_{0<|\mathbf{k}|\leq N}\frac{1}{|\mathbf{k}|^2}\right)^{\frac{1}{2}} + 2c_P(\|\widetilde{\mathbf{w}}_2\||\eta|+\|\widetilde{\mathbf{v}}_1\||\zeta|)|\psi^v-\psi^w|\left(\sum_{0<|\mathbf{k}|\leq N}\frac{1}{|\mathbf{k}|^4}\right)^{\frac{1}{2}}.$$

We prove the following estimates:

$$\sum_{0<|\mathbf{k}|\leq N}\frac{1}{|\mathbf{k}|^2}=2+\sum_{1<|\mathbf{k}|\leq N}\frac{1}{|\mathbf{k}|^2}\leq 2+\int_0^{\frac{\pi}{2}}\int_1^N\frac{1}{r^2}drd\theta\leq 2+\frac{\pi}{2}\leq 4,$$

$$\sum_{0<|\mathbf{k}|\leq N}\frac{1}{|\mathbf{k}|^4}=2+\sum_{1<|\mathbf{k}|\leq N}\frac{1}{|\mathbf{k}|^4}\leq 2+\int_0^{\frac{\pi}{2}}\int_1^N\frac{1}{r^4}drd\theta\leq 2+\frac{\pi}{6}\leq 4.$$

Thus, using lemmas 3.1 and 3.2, we have the inequalities:

$$\begin{aligned}|(\widetilde{\alpha}_1-\widetilde{\alpha}_2)+(\widetilde{\beta}_1-\widetilde{\beta}_2)||\psi^v+\psi^w| &\leq 2c_P(|\widetilde{\mathbf{w}}_2|+|\widetilde{\mathbf{v}}_1|)(|\eta|+|\zeta|)\\ &\leq 2\sqrt{2}c_P M_H(\phi^v,\phi^w)(|\eta|+|\zeta|),\end{aligned}$$

$$\begin{aligned}|(\widetilde{\alpha}_1-\widetilde{\alpha}_2)-(\widetilde{\beta}_1-\widetilde{\beta}_2)||\psi^v-\psi^w| &\leq 2c_P(|\widetilde{\mathbf{w}}_2|+|\widetilde{\mathbf{v}}_1|)(|\eta|+|\zeta|)\\ &\quad + 4c_P(\|\widetilde{\mathbf{w}}_2\|+\|\widetilde{\mathbf{v}}_1\|)(|\eta|+|\zeta|)\\ &\leq 2\sqrt{2}c_P(M_H(\phi^v,\phi^w)+2M_V(\phi^v,\phi^w))(|\eta|+|\zeta|)\end{aligned}$$

Thanks to corollary 3.1, we have the estimate holding for any $p\in[0,1]$:

$$\begin{aligned}&\|\eta\|_{C_b(H)}+\|\zeta\|_{C_b(H)}\\ &\quad\leq\frac{2\sqrt{2}c_P}{\sqrt{\nu_{\min}}N}(\nu'_{\max})^{\frac{1}{2}-\frac{p}{2}}\left(|\widetilde{\alpha}_1-\widetilde{\alpha}_2|^{2-p}+|\widetilde{\beta}_1-\widetilde{\beta}_2|^{2-p}\right)^{\frac{1}{2}}\left(\|\overline{\mathbf{w}}\|^2_{C_b(V)}+\|\overline{\mathbf{v}}\|^2_{C_b(V)}\right)^{\frac{1}{2}}\\ &\quad\leq\frac{2\sqrt{2}c_P}{\sqrt{\nu_{\min}}N}(\nu'_{\max})^{\frac{1}{2}-\frac{p}{2}}\left(|\widetilde{\alpha}_1-\widetilde{\alpha}_2|^{2-p}+|\widetilde{\beta}_1-\widetilde{\beta}_2|^{2-p}\right)^{\frac{1}{2}}M_V(\phi^v,\phi^w).\end{aligned}$$

Thus we have proved the inequalities for any $p\in[0,1]$:

$$\begin{aligned}&|(\widetilde{\alpha}_1-\widetilde{\alpha}_2)+(\widetilde{\beta}_1-\widetilde{\beta}_2)||\psi^v+\psi^w|\\ &\quad\leq\frac{8c_P^2}{\sqrt{\nu_{\min}}N}(\nu'_{\max})^{\frac{1}{2}-\frac{p}{2}}M_H(\phi^v,\phi^w)M_V(\phi^v,\phi^w)\left(|\widetilde{\alpha}_1-\widetilde{\alpha}_2|^{2-p}+|\widetilde{\beta}_1-\widetilde{\beta}_2|^{2-p}\right)^{\frac{1}{2}},\end{aligned}$$

$$\begin{aligned}&|(\widetilde{\alpha}_1-\widetilde{\alpha}_2)-(\widetilde{\beta}_1-\widetilde{\beta}_2)||\psi^v-\psi^w|\\ &\quad\leq\frac{8c_P^2}{\sqrt{\nu_{\min}}N}(\nu'_{\max})^{\frac{1}{2}-\frac{p}{2}}(M_H(\phi^v,\phi^w)+2M_V(\phi^v,\phi^w))M_V(\phi^v,\phi^w)\left(|\widetilde{\alpha}_1-\widetilde{\alpha}_2|^{2-p}+|\widetilde{\beta}_1-\widetilde{\beta}_2|^{2-p}\right)^{\frac{1}{2}}.\end{aligned}$$

Proof of lemma 3.3 First, note that:

$$\begin{aligned}\frac{\overline{\alpha}-\overline{\beta}}{2}&\geq\gamma\\ &\geq\min\{\alpha_1,\alpha_2\}-\min\{\beta_1,\beta_2\}\\ &\geq\nu_{\min}\end{aligned}$$

Let $\eta = \widetilde{\mathbf{v}}_1 - \widetilde{\mathbf{v}}_2$ and $\zeta = \widetilde{\mathbf{w}}_1 - \widetilde{\mathbf{w}}_2$. They verify the differential equation

$$\partial_t \eta + \mathcal{B}(\zeta, \overline{\mathbf{v}}) + \mathcal{B}(\overline{\mathbf{w}}, \eta) + \overline{\alpha}\mathcal{A}\eta + \overline{\beta}\mathcal{A}\zeta = -(\widetilde{\alpha}_1 - \widetilde{\alpha}_2)\mathcal{A}\overline{\mathbf{v}} - (\widetilde{\beta}_1 - \widetilde{\beta}_2)\mathcal{A}\overline{\mathbf{w}} + \mu(\phi_1^v - \phi_v^2 - P_N\eta),$$
$$\partial_t \zeta + \mathcal{B}(\eta, \overline{\mathbf{w}}) + \mathcal{B}(\overline{\mathbf{v}}, \zeta) + \overline{\alpha}\mathcal{A}\zeta + \overline{\beta}\mathcal{A}\eta = -(\widetilde{\alpha}_1 - \widetilde{\alpha}_2)\mathcal{A}\overline{\mathbf{w}} - (\widetilde{\beta}_1 - \widetilde{\beta}_2)\mathcal{A}\overline{\mathbf{v}} + \mu(\phi_1^w - \phi_2^w - P_N\zeta).$$

We test the first equation with η:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\eta|^2 + \overline{\alpha}\|\eta\|^2 + \mu|P_N\eta|^2 =& -\overline{\beta}\langle \mathcal{A}\zeta, \eta\rangle - (\widetilde{\alpha}_1 - \widetilde{\alpha}_2)\langle \mathcal{A}\overline{\mathbf{v}}, \eta\rangle - (\widetilde{\beta}_1 - \widetilde{\beta}_2)\langle \mathcal{A}\overline{\mathbf{w}}, \eta\rangle \\ &- \langle \mathcal{B}(\zeta, \overline{\mathbf{v}}), \eta\rangle \\ \le& \frac{\overline{\beta}}{2}(\|\zeta\|^2 + \|\eta\|^2) + |\widetilde{\alpha}_1 - \widetilde{\alpha}_2|\|\overline{\mathbf{v}}\|\|\eta\| + |\widetilde{\beta}_1 - \widetilde{\beta}_2|\|\overline{\mathbf{w}}\|\|\eta\| \\ &+ \mu|\phi_1^v - \phi_2^v||\eta| + \|\overline{\mathbf{v}}\||\zeta|^{\frac{1}{2}}\|\zeta\|^{\frac{1}{2}}|\eta|^{\frac{1}{2}}\|\eta\|^{\frac{1}{2}}\end{aligned}$$

We now make use of Young's inequality (11) on each terms. For some positive constants $\varepsilon_1, \varepsilon_2 > 0$:

$$\begin{aligned}|\widetilde{\alpha}_1 - \widetilde{\alpha}_2|\|\overline{\mathbf{v}}\|\|\eta\| \le& \frac{|\widetilde{\alpha}_1 - \widetilde{\alpha}_2|}{2}\left(\varepsilon_1\|\eta\|^2 + \frac{1}{\varepsilon_1}\|\overline{\mathbf{v}}\|^2\right), \\ |\widetilde{\beta}_1 - \widetilde{\beta}_2|\|\overline{\mathbf{w}}\|\|\eta\| \le& \frac{|\widetilde{\beta}_1 - \widetilde{\beta}_2|}{2}\left(\varepsilon_2\|\eta\|^2 + \frac{1}{\varepsilon_2}\|\overline{\mathbf{w}}\|^2\right).\end{aligned}$$

We also use Young's inequality and the identity $|\eta|^2 = |\eta - P_N\eta|^2 + |P_N\eta|^2$ to prove that for any $\varepsilon_3 > 0$:

$$\begin{aligned}\mu|\phi_1^v - \phi_2^v||\eta| \le& \frac{\mu}{2\varepsilon_3}|\phi_1^v - \phi_2^v|^2 + \frac{\mu\varepsilon_3}{2}|\eta|^2 \\ \le& \frac{\mu}{2\varepsilon_3}|\phi_1^v - \phi_2^v|^2 + \frac{\mu\varepsilon_3 c_P^2}{2N^2}\|\eta\|^2 + \frac{\mu\varepsilon_3}{2}|P_N\eta|^2.\end{aligned}$$

Thus gathering all these inequalities and rearranging the terms, we obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\eta|^2 + \left(\overline{\alpha} - \frac{\overline{\beta}}{2} - \frac{\varepsilon_1}{2}|\widetilde{\alpha}_1 - \widetilde{\alpha}_2| - \frac{\varepsilon_2}{2}|\widetilde{\beta}_1 - \widetilde{\beta}_2| - \frac{\mu\varepsilon_3 c_P^2}{2N^2}\right)\|\eta\|^2 - \frac{\overline{\beta}}{2}\|\zeta\|^2 \\ + \mu\left(1 - \frac{\varepsilon_3}{2}\right)|P_N\eta|^2 \le \frac{|\widetilde{\alpha}_1 - \widetilde{\alpha}_2|}{2\varepsilon_1}\|\overline{\mathbf{v}}\|^2 + \frac{|\widetilde{\beta}_1 - \widetilde{\beta}_2|}{2\varepsilon_2}\|\overline{\mathbf{w}}\|^2 + \frac{\mu}{2\varepsilon_3}|\phi_1^v - \phi_2^v|^2 \\ + \|\overline{\mathbf{v}}\||\zeta|^{\frac{1}{2}}\|\zeta\|^{\frac{1}{2}}|\eta|^{\frac{1}{2}}\|\eta\|^{\frac{1}{2}}.\end{aligned}$$

Similarly, one proves:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\zeta|^2 + \left(\overline{\alpha} - \frac{\overline{\beta}}{2} - \frac{\varepsilon_1}{2}|\widetilde{\alpha}_1 - \widetilde{\alpha}_2| - \frac{\varepsilon_2}{2}|\widetilde{\beta}_1 - \widetilde{\beta}_2| - \frac{\mu\varepsilon_3 c_P^2}{2N^2}\right)\|\zeta\|^2 - \frac{\overline{\beta}}{2}\|\eta\|^2 \\ + \mu\left(1 - \frac{\varepsilon_3}{2}\right)|P_N\zeta|^2 \le \frac{|\widetilde{\alpha}_1 - \widetilde{\alpha}_2|}{2\varepsilon_1}\|\overline{\mathbf{w}}\|^2 + \frac{|\widetilde{\beta}_1 - \widetilde{\beta}_2|}{2\varepsilon_2}\|\overline{\mathbf{v}}\|^2 + \frac{\mu}{2\varepsilon_3}|\phi_1^w - \phi_2^w|^2 \\ + \|\overline{\mathbf{w}}\||\zeta|^{\frac{1}{2}}\|\zeta\|^{\frac{1}{2}}|\eta|^{\frac{1}{2}}\|\eta\|^{\frac{1}{2}}.\end{aligned}$$

Summing the two inequalities, we get:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|\eta|^2+|\zeta|^2)+&\left(\overline{\alpha}-\overline{\beta}-\frac{\varepsilon_1}{2}|\widetilde{\alpha}_1-\widetilde{\alpha}_2|-\frac{\varepsilon_2}{2}|\widetilde{\beta}_1-\widetilde{\beta}_2|-\frac{\mu\varepsilon_3 c_P^2}{2N^2}\right)(\|\eta\|^2+\|\zeta\|^2)\\&+\mu\left(1-\frac{\varepsilon_3}{2}\right)(|P_N\eta|^2+|P_N\zeta|^2)\\&\le\left(\frac{|\widetilde{\alpha}_1-\widetilde{\alpha}_2|}{2\varepsilon_1}+\frac{|\widetilde{\beta}_1-\widetilde{\beta}_2|}{2\varepsilon_2}\right)(\|\overline{\mathbf{w}}\|^2+\|\overline{\mathbf{v}}\|^2)\\&+\frac{\mu}{2\varepsilon_3}(|\phi_1^v-\phi_2^v|^2+|\phi_1^w-\phi_2^w|^2)\\&+|\zeta|^{\frac12}\|\zeta\|^{\frac12}|\eta|^{\frac12}\|\eta\|^{\frac12}(\|\overline{\mathbf{v}}\|+\|\overline{\mathbf{w}}\|).\end{aligned}$$

We now choose $\varepsilon_1=\overline{\alpha}^{1-p}|\widetilde{\alpha}_1-\widetilde{\alpha}_2|^{p-1}$ and $\varepsilon_2=\overline{\beta}^{1-p}|\widetilde{\beta}_1-\widetilde{\beta}_2|^{p-1}$, $\varepsilon_3=\frac{N^2}{\mu c_P^2}\gamma$. After some calculations, the inequality becomes:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|\eta|^2+|\zeta|^2)+&\frac{\gamma}{2}(\|\eta\|^2+\|\zeta\|^2)+\left(\mu-\frac{N^2}{2c_P^2}\gamma\right)(|P_N\eta|^2+|P_N\zeta|^2)\\&\le\frac{1}{2}\left(\overline{\alpha}^{1-p}|\widetilde{\alpha}_1-\widetilde{\alpha}_2|^{2-p}+\overline{\beta}^{1-p}|\widetilde{\beta}_1-\widetilde{\beta}_2|^{2-p}\right)(\|\overline{\mathbf{w}}\|^2+\|\overline{\mathbf{v}}\|^2)\\&+\frac{\mu^2c_P^2}{2\gamma N^2}(|\phi_1^v-\phi_2^v|^2+|\phi_1^w-\phi_2^w|^2)+|\zeta|^{\frac12}\|\zeta\|^{\frac12}|\eta|^{\frac12}\|\eta\|^{\frac12}(\|\overline{\mathbf{v}}\|+\|\overline{\mathbf{w}}\|).\end{aligned}$$

We now examine the last term on the right hand side. First, using Young's inequality, we remark that $|\zeta|^{\frac12}\|\zeta\|^{\frac12}|\eta|^{\frac12}\|\eta\|^{\frac12}\le\frac12(|\zeta|\|\zeta\|+|\eta|\|\eta\|)$. Then, using (2), we prove that:

$$\begin{aligned}|\eta|\|\eta\|&\le\|\eta\|(|\eta-P_N\eta|+|P_N\eta|)\\&\le\|\eta\||P_N\eta|+\frac{c_P}{N}\|\eta\|^2\\&\le\frac{3}{2}\frac{c_P}{N}\|\eta\|^2+\frac{N}{2c_P}|P_N\eta|^2.\end{aligned}$$

The same inequality holds for ζ. Thus:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|\eta|^2+|\zeta|^2)+&\frac{1}{2}\left(\gamma-3\frac{c_P}{N}(\|\overline{\mathbf{v}}\|+\|\overline{\mathbf{w}}\|)\right)(\|\eta\|^2+\|\zeta\|^2)\\&+\left(\mu-\frac{N^2}{2c_P^2}\gamma-\frac{N}{2c_P}(\|\overline{\mathbf{v}}\|+\|\overline{\mathbf{w}}\|)\right)(|P_N\eta|^2+|P_N\zeta|^2)\\&\le\frac{1}{2}\left(\overline{\alpha}^{1-p}|\widetilde{\alpha}_1-\widetilde{\alpha}_2|^{2-p}+\overline{\beta}^{1-p}|\widetilde{\beta}_1-\widetilde{\beta}_2|^{2-p}\right)(\|\overline{\mathbf{w}}\|^2+\|\overline{\mathbf{v}}\|^2)\\&+\frac{\mu^2c_P^2}{2\gamma N^2}(|\phi_1^v-\phi_2^v|^2+|\phi_1^w-\phi_2^w|^2).\end{aligned}$$

Using (2), we prove:

$$\|\eta\|^2\ge\frac{N^2}{c_P^2}|\eta-P_N\eta|^2=\frac{N^2}{c_P^2}(|\eta|^2-|P_N\eta|^2).$$

The differential inequality then becomes:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|\eta|^2+|\zeta|^2)+&\frac{N^2}{2c_P^2}\left(\gamma-3\frac{c_P}{N}(\|\overline{\mathbf{v}}\|+\|\overline{\mathbf{w}}\|)\right)(|\eta|^2+|\zeta|^2)\\&+\left(\mu-\frac{N^2}{c_P^2}\gamma+\frac{N}{c_P}(\|\overline{\mathbf{v}}\|+\|\overline{\mathbf{w}}\|)\right)(|P_N\eta|^2+|P_N\zeta|^2)\\\leq&\frac{1}{2}\left(\overline{\alpha}^{1-p}|\widetilde{\alpha}_1-\widetilde{\alpha}_2|^{2-p}+\overline{\beta}^{1-p}|\widetilde{\beta}_1-\widetilde{\beta}_2|^{2-p}\right)(\|\overline{\mathbf{w}}\|^2+\|\overline{\mathbf{v}}\|^2)\\&+\frac{\mu^2c_P^2}{2\gamma N^2}(|\phi_1^v-\phi_2^v|^2+|\phi_1^w-\phi_2^w|^2).\end{aligned}$$

But since we supposed that $\mu\geq\frac{\gamma N^2}{c_P^2}$, we have that $\mu-\frac{N^2}{c_P^2}\gamma+\frac{N}{c_P}(\|\overline{\mathbf{v}}\|+\|\overline{\mathbf{w}}\|)\geq 0$. Thus, we can drop the last term in the left hand side, and the inequality becomes:

$$\begin{aligned}\frac{d}{dt}(|\eta|^2+|\zeta|^2)+&\frac{N^2}{c_P^2}\left(\gamma-3\frac{c_P}{N}(\|\overline{\mathbf{v}}\|+\|\overline{\mathbf{w}}\|)\right)(|\eta|^2+|\zeta|^2)\\\leq&\left(\overline{\alpha}^{1-p}|\widetilde{\alpha}_1-\widetilde{\alpha}_2|^{2-p}+\overline{\beta}^{1-p}|\widetilde{\beta}_1-\widetilde{\beta}_2|^{2-p}\right)(\|\overline{\mathbf{w}}\|^2+\|\overline{\mathbf{v}}\|^2)\\&+\frac{\mu^2c_P^2}{\gamma N^2}(|\phi_1^v-\phi_2^v|^2+|\phi_1^w-\phi_2^w|^2)\end{aligned}$$

One can easily prove that for all $x,y\geq 0$, $2(x^2+y^2)^{\frac{1}{2}}\geq x+y$. Since we suppose that $N\geq\frac{8c_P}{\gamma}M_V(\phi_i^v,\phi_i^w)$ for $i=1,2$, and using (14), we have:

$$N\geq\frac{4c_P}{\gamma}\frac{\|\widetilde{\mathbf{v}}_1\|+\|\widetilde{\mathbf{v}}_2\|+\|\widetilde{\mathbf{w}}_1\|+\|\widetilde{\mathbf{w}}_2\|}{2}\geq\frac{4c_P}{\gamma}(\|\overline{\mathbf{v}}\|+\|\overline{\mathbf{w}}\|).$$

Thus, $\gamma-3\frac{c_P}{N}(\|\overline{\mathbf{v}}\|+\|\overline{\mathbf{w}}\|)\geq\gamma-\frac{3\gamma}{4}=\frac{\gamma}{4}>0$. This proves the differential inequality:

$$\begin{aligned}\frac{d}{dt}(|\eta|^2+|\zeta|^2)+&\frac{\gamma N^2}{4c_P^2}(|\eta|^2+|\zeta|^2)\\\leq&\left(\overline{\alpha}^{1-p}|\widetilde{\alpha}_1-\widetilde{\alpha}_2|^{2-p}+\overline{\beta}^{1-p}|\widetilde{\beta}_1-\widetilde{\beta}_2|^{2-p}\right)(\|\overline{\mathbf{w}}\|^2_{C_b(V)}+\|\overline{\mathbf{v}}\|^2_{C_b(V)})\\&+\frac{\mu^2c_P^2}{\gamma N^2}(\|\phi_1^v-\phi_2^v\|^2_{C_b(H)}+\|\phi_1^w-\phi_2^w\|^2_{C_b(H)}).\end{aligned}$$

Denote $\mathfrak{p}=\left(\overline{\alpha}^{1-p}|\widetilde{\alpha}_1-\widetilde{\alpha}_2|^{2-p}+\overline{\beta}^{1-p}|\widetilde{\beta}_1-\widetilde{\beta}_2|^{2-p}\right)(\|\overline{\mathbf{w}}\|^2_{C_b(V)}+\|\overline{\mathbf{v}}\|^2_{C_b(V)})+\frac{\mu^2c_P^2}{\gamma N^2}(\|\phi_1^v-\phi_2^v\|^2_{C_b(H)}+\|\phi_1^w-\phi_2^w\|^2_{C_b(H)})$. Using Grönwall's lemma we prove that, for all $-\infty<s<t<+\infty$:

$$|\eta(t)|^2+|\zeta(t)|^2\leq\exp\left(-\frac{\gamma N^2}{4c_P^2}(t-s)\right)(|\eta(s)|^2+|\zeta(s)|^2)+\frac{4c_P^2}{\gamma N^2}\mathfrak{p}.$$

Taking the limit as $s\to-\infty$ proves the result.

Proof of lemma 4.2 Let $\eta=\mathbf{v}_1-\mathbf{v}_2$, $\zeta=\mathbf{w}_1-\mathbf{w}_2$, $\psi^v=P_N\mathbf{v}_1$ and $\psi^w=P_N\mathbf{w}_1$. The equations for η and ζ are:

$$\begin{aligned}&\partial_t\eta+(\alpha_1-\alpha_2)\mathcal{A}\mathbf{v}_1+\alpha_2\mathcal{A}\eta+(\beta_1-\beta_2)\mathcal{A}\mathbf{w}_1+\beta_2\mathcal{A}\zeta+\mathcal{B}(\mathbf{v}_1,\zeta)+\mathcal{B}(\eta,\mathbf{w}_2)+\mu P_N\eta=0,\\&\partial_t\zeta+(\alpha_1-\alpha_2)\mathcal{A}\mathbf{w}_1+\alpha_2\mathcal{A}\zeta+(\beta_1-\beta_2)\mathcal{A}\mathbf{v}_1+\beta_2\mathcal{A}\eta+\mathcal{B}(\mathbf{w}_1,\eta)+\mathcal{B}(\zeta,\mathbf{v}_2)+\mu P_N\zeta=0.\end{aligned}\qquad(23)$$

We test the first equation of (23) with $\mathcal{A}^{-1}\psi^v$. Using also the facts that $\mathcal{A}^{-1}$ is symmetric, P_N is a self-adjoint projection, P_N commutes with $\mathcal{A}^{-1}$ and $\frac{d}{dt}\langle \mathcal{A}^{-1}P_N\eta, \psi^v\rangle = \langle \partial_t\eta, \mathcal{A}^{-1}\psi^v\rangle + \langle P_N\eta, \mathcal{A}^{-1}\partial_t\psi^v\rangle$, we have:

$$\begin{aligned}\frac{d}{dt}\langle \mathcal{A}^{-1}P_N\eta, \psi^v\rangle + &\mu\langle \mathcal{A}^{-1}P_N\eta, \psi^v\rangle - \langle \mathcal{A}^{-1}P_N\eta, \partial_t\psi^v\rangle + (\alpha_1-\alpha_2)|\psi^v|^2\\ &+\alpha_2\langle \eta, \psi^v\rangle + (\beta_1-\beta_2)\langle \psi^v, \psi^w\rangle + \beta_2\langle \zeta, \psi^v\rangle\\ &+\langle \mathcal{B}(\mathbf{v}_1, \zeta) + \mathcal{B}(\eta, \mathbf{w}_2), \mathcal{A}^{-1}\psi^v\rangle = 0.\end{aligned}$$

Denote $\rho^v_\eta = \langle \mathcal{A}^{-1}P_N\eta, \psi^v\rangle$. The equation becomes:

$$\begin{aligned}\dot{\rho^v_\eta} + &\mu\rho^v_\eta - \langle \mathcal{A}^{-1}P_N\eta, \partial_t\psi^v\rangle + (\alpha_1-\alpha_2)|\psi^v|^2\\ &+\alpha_2\langle \eta, \psi^v\rangle + (\beta_1-\beta_2)\langle \psi^v, \psi^w\rangle + \beta_2\langle \zeta, \psi^v\rangle\\ &+\langle \mathcal{B}(\mathbf{v}_1, \zeta) + \mathcal{B}(\eta, \mathbf{w}_2), \mathcal{A}^{-1}\psi^v\rangle = 0.\end{aligned}$$

We multiply the equation by $e^{-\mu(t-\tau)}$ and time-intergate over $[s,t]$:

$$\begin{aligned}\rho^v_\eta(t) - &e^{-\mu(t-s)}\rho^v_\eta(s) + (\alpha_1-\alpha_2)\int_s^t e^{-\mu(t-\tau)}|\psi^v(\tau)|^2 d\tau\\ &-\int_s^t e^{-\mu(t-\tau)}\langle \mathcal{A}^{-1}P_N\eta, \partial_t\psi^v\rangle d\tau + \alpha_2\int_s^t e^{-\mu(t-\tau)}\langle \eta, \psi^v\rangle d\tau\\ &+(\beta_1-\beta_2)\int_s^t e^{-\mu(t-\tau)}\langle \psi^v, \psi^w\rangle d\tau + \beta_2\int_s^t e^{-\mu(t-\tau)}\langle \zeta, \psi^v\rangle d\tau\\ &+\int_s^t e^{-\mu(t-\tau)}(\langle \mathcal{B}(\mathbf{v}_1, \zeta) + \mathcal{B}(\eta, \mathbf{w}_2), \mathcal{A}^{-1}\psi^v\rangle) d\tau = 0.\end{aligned} \tag{24}$$

Denote $\rho^w_\zeta = \langle \mathcal{A}^{-1}P_N\zeta, \psi^w\rangle$. With similar calculations, one proves that:

$$\begin{aligned}\rho^w_\zeta(t) - &e^{-\mu(t-s)}\rho^w_\zeta(s) + (\alpha_1-\alpha_2)\int_s^t e^{-\mu(t-\tau)}|\psi^w(\tau)|^2 d\tau\\ &-\int_s^t e^{-\mu(t-\tau)}\langle \mathcal{A}^{-1}P_N\zeta, \partial_t\psi^w\rangle d\tau + \alpha_2\int_s^t e^{-\mu(t-\tau)}\langle \zeta, \psi^w\rangle d\tau\\ &+(\beta_1-\beta_2)\int_s^t e^{-\mu(t-\tau)}\langle \psi^v, \psi^w\rangle d\tau + \beta_2\int_s^t e^{-\mu(t-\tau)}\langle \eta, \psi^w\rangle d\tau\\ &+\int_s^t e^{-\mu(t-\tau)}(\langle \mathcal{B}(\mathbf{w}_1, \eta) + \mathcal{B}(\zeta, \mathbf{v}_2), \mathcal{A}^{-1}\psi^w\rangle) d\tau = 0.\end{aligned} \tag{25}$$

Doing the difference of (24) and (25) and similar calculations, one proves that:

$$
\begin{aligned}
\rho_\eta^v(t) - \rho_\zeta^w(t) - & e^{-\mu(t-s)}(\rho_\eta^v(s) - \rho_\zeta^w(s)) \\
& +(\alpha_1 - \alpha_2)\int_s^t e^{-\mu(t-\tau)}(|\psi^v(\tau)|^2 - |\psi^w(\tau)|^2)d\tau \\
& -\int_s^t e^{-\mu(t-\tau)}(\langle P_N\eta, \mathcal{A}^{-1}\partial_t\psi^v\rangle - \langle P_N\zeta, \mathcal{A}^{-1}\partial_t\psi^w\rangle)d\tau \\
& +\alpha_2\int_s^t e^{-\mu(t-\tau)}(\langle \eta, \psi^v\rangle - \langle \zeta, \psi^w\rangle)d\tau \\
& +\beta_2\int_s^t e^{-\mu(t-\tau)}(\langle \zeta, \psi^v\rangle - \langle \eta, \psi^w\rangle)d\tau \\
& +\int_s^t e^{-\mu(t-\tau)}(\langle \mathcal{B}(\mathbf{v}_1, \zeta) + \mathcal{B}(\eta, \mathbf{w}_2), \mathcal{A}^{-1}\psi^v\rangle)d\tau \\
& -\int_s^t e^{-\mu(t-\tau)}(\langle \mathcal{B}(\mathbf{w}_1, \eta) + \mathcal{B}(\zeta, \mathbf{v}_2), \mathcal{A}^{-1}\psi^w\rangle)d\tau = 0.
\end{aligned} \tag{26}
$$

Thus:

$$
\begin{aligned}
|\alpha_1 - \alpha_2|\mu^{-1}(1 - e^{-\mu(t-s)}) & \inf_{[s,t]} \left||\psi^v|^2 - |\psi^w|^2\right| \\
\leq & |\rho_\eta^v(t) - \rho_\zeta^w(t)| + e^{-\mu(t-s)}|\rho_\eta^v(s) - \rho_\zeta^w(s)| \\
& + \mu^{-1}(1 - e^{-\mu(t-s)}) \sup_{[s,t]} \left(|\mathcal{A}^{-1}\partial_t\psi^v| + \alpha_2|\psi^v| + \beta_2|\psi^w|\right) \|P_N\eta\|_{C_b(H)} \\
& + \mu^{-1}(1 - e^{-\mu(t-s)}) \sup_{[s,t]} \left(|\mathcal{A}^{-1}\partial_t\psi^w| + \alpha_2|\psi^w| + \beta_2|\psi^v|\right) \|P_N\zeta\|_{C_b(H)} \\
& + \mu^{-1}(1 - e^{-\mu(t-s)}) \sup_{[s,t]} \left(\|\mathbf{v}_1\|_{L^4(\Omega)}\|\mathcal{A}^{-\frac{1}{2}}\psi^v\|_{L^4(\Omega)} + \|\mathbf{v}_2\|_{L^4(\Omega)}\|\mathcal{A}^{-\frac{1}{2}}\psi^w\|_{L^4(\Omega)}\right) \|\eta\|_{C_b(H)} \\
& + \mu^{-1}(1 - e^{-\mu(t-s)}) \sup_{[s,t]} \left(\|\mathbf{w}_1\|_{L^4(\Omega)}\|\mathcal{A}^{-\frac{1}{2}}\psi^w\|_{L^4(\Omega)} + \|\mathbf{w}_2\|_{L^4(\Omega)}\|\mathcal{A}^{-\frac{1}{2}}\psi^v\|_{L^4(\Omega)}\right) \|\zeta\|_{C_b(H)}.
\end{aligned}
$$

Note the following estimates:

- $|\rho_\eta^v - \rho_\zeta^w| \leq \|P_N\eta\|_{C_b(H)}\|\mathcal{A}^{-1}\psi^v\|_{C_b(H)} + \|P_N\zeta\|_{C_b(H)}\ \|\mathcal{A}^{-1}\psi^w\|_{C_b(H)} < \infty$.
- For $i = 1, 2$, using the continuous inclusion of $H^1(\Omega)$ into $L^4(\Omega)$, $\|\mathbf{v}_i\|_{L^4(\Omega)} \leq c_L^{\frac{1}{2}}\|\mathbf{v}_i\|_{C_b(H)}^{\frac{1}{2}}\|\mathbf{v}_i\|_{C_b(V)}^{\frac{1}{2}} \leq c_L^{\frac{1}{2}}M_H(\phi^v, \phi^w)^{\frac{1}{2}}M_V(\phi^v, \phi^w)^{\frac{1}{2}}$. Similarly, $\|\mathbf{w}_i\|_{L^4(\Omega)} \leq c_L^{\frac{1}{2}}M_H(\phi^v, \phi^w)^{\frac{1}{2}}M_V(\phi^v, \phi^w)^{\frac{1}{2}}$.
- Using also Poincaré's inequality, $\|\mathcal{A}^{-\frac{1}{2}}\psi^v\|_{L^4(\Omega)} \leq c_L^{\frac{1}{2}}|\mathcal{A}^{-\frac{1}{2}}\psi^v|^{\frac{1}{2}}|\psi^v|^{\frac{1}{2}} \leq c_P c_L^{\frac{1}{2}}\|\mathbf{v}_1\|_{C_b(H)}^{\frac{1}{2}}\|\mathbf{v}_1\|_{C_b(V)}^{\frac{1}{2}} \leq c_P c_L^{\frac{1}{2}}M_H(\phi^v, \phi^w)^{\frac{1}{2}}M_V(\phi^v, \phi^w)^{\frac{1}{2}}$. Similarly, $\|\mathcal{A}^{-\frac{1}{2}}\psi^w\|_{L^4(\Omega)} \leq c_P c_L^{\frac{1}{2}}M_H(\phi^v, \phi^w)^{\frac{1}{2}}M_V(\phi^v, \phi^w)^{\frac{1}{2}}$.

Define $\delta = e^{-\frac{\mu(t-s)}{N^2}}$. The previous inequality then becomes, thanks to corollary 3.1:

$$\begin{aligned}
&|\alpha_1-\alpha_2|\inf_{[s,t]}\left||\psi^v|^2-|\psi^w|^2\right| \\
&\quad\le \mu\frac{1+\delta^{N^2}}{1-\delta^{N^2}}\left(\|P_N\eta\|_{C_b(H)}\|\mathcal{A}^{-1}\psi^v\|_{C_b(H)}+\|P_N\zeta\|_{C_b(H)}\|\mathcal{A}^{-1}\psi^w\|_{C_b(H)}\right) \\
&\quad\quad+\left(\|\mathcal{A}^{-1}\partial_t\psi^v\|_{C_b(H)}+\alpha_2\|\psi^v\|_{C_b(H)}+\beta_2\|\psi^w\|_{C_b(H)}\right)\|P_N\eta\|_{C_b(H)} \\
&\quad\quad+\left(\|\mathcal{A}^{-1}\partial_t\psi^w\|_{C_b(H)}+\alpha_2\|\psi^w\|_{C_b(H)}+\beta_2\|\psi^v\|_{C_b(H)}\right)\|P_N\zeta\|_{C_b(H)} \\
&\quad\quad+2c_Pc_LM_H(\phi^v,\phi^w)M_V(\phi^v,\phi^w)(\|\eta\|_{C_b(H)}+\|\zeta\|_{C_b(H)}) \\
&\quad\le \widetilde{M}\left(\|P_N\eta\|_{C_b(H)}+\|P_N\zeta\|_{C_b(H)}\right) \\
&\quad\quad+\frac{4\sqrt{2}c_P^2\sqrt{\nu'_{\max}}}{\sqrt{\nu_{\min}}N}c_LM_H(\phi^v,\phi^w)M_V(\phi^v,\phi^w)^2(|\alpha_1-\alpha_2|+|\beta_1-\beta_2|),
\end{aligned} \tag{27}$$

where $\widetilde{M}=\|\mathcal{A}^{-1}\partial_t\psi^v\|_{C_b(H)}+\|\mathcal{A}^{-1}\partial_t\psi^w\|_{C_b(H)}+\nu'_{\max}(\|\psi^v\|_{C_b(H)}+\|\psi^w\|_{C_b(H)})+\mu\frac{1+\delta^{N^2}}{1-\delta^{N^2}}\left(\|\mathcal{A}^{-1}\psi^v\|_{C_b(H)}+\|\mathcal{A}^{-1}\psi^w\|_{C_b(H)}\right)$

Similarly, we test the first equation of (23) with $\mathcal{A}^{-1}\psi^w$. Denoting $\rho_\eta^w=\langle\mathcal{A}^{-1}P_N\eta,\psi^w\rangle$, we have:

$$\begin{aligned}
&\dot{\rho_\eta^w}+\mu\rho_\eta^w-\langle\mathcal{A}^{-1}P_N\eta,\partial_t\psi^w\rangle+(\alpha_1-\alpha_2)\langle\psi^v,\psi^w\rangle \\
&\quad+\alpha_2\langle\eta,\psi^w\rangle+(\beta_1-\beta_2)|\psi^w|^2+\beta_2\langle\zeta,\psi^w\rangle \\
&\quad+\langle\mathcal{B}(\mathbf{v}_1,\zeta)+\mathcal{B}(\eta,\mathbf{w}_2),\mathcal{A}^{-1}\psi^w\rangle=0.
\end{aligned}$$

We multiply the equation by $e^{-\mu(t-\tau)}$ and time-intergate over $[s,t]$:

$$\begin{aligned}
&\rho_\eta^w(t)-e^{-\mu(t-s)}\rho_\eta^w(s)+(\alpha_1-\alpha_2)\int_s^t e^{-\mu(t-\tau)}\langle\psi^v,\psi^w\rangle d\tau \\
&\quad-\int_s^t e^{-\mu(t-\tau)}\langle\mathcal{A}^{-1}P_N\eta,\partial_t\psi^w\rangle d\tau+\alpha_2\int_s^t e^{-\mu(t-\tau)}\langle\eta,\psi^w\rangle d\tau \\
&\quad+(\beta_1-\beta_2)\int_s^t e^{-\mu(t-\tau)}|\psi^w(\tau)|^2d\tau+\beta_2\int_s^t e^{-\mu(t-\tau)}\langle\zeta,\psi^w\rangle d\tau \\
&\quad+\int_s^t e^{-\mu(t-\tau)}(\langle\mathcal{B}(\mathbf{v}_1,\zeta)+\mathcal{B}(\eta,\mathbf{w}_2),\mathcal{A}^{-1}\psi^w\rangle)d\tau=0.
\end{aligned} \tag{28}$$

Denote $\rho_\zeta^v=\langle\mathcal{A}^{-1}P_N\zeta,\psi^v\rangle$. With similar arguments, one proves that:

$$\begin{aligned}
&\rho_\zeta^v(t)-e^{-\mu(t-s)}\rho_\zeta^v(s)+(\alpha_1-\alpha_2)\int_s^t e^{-\mu(t-\tau)}\langle\psi^v,\psi^w\rangle d\tau \\
&\quad-\int_s^t e^{-\mu(t-\tau)}\langle\mathcal{A}^{-1}P_N\zeta,\partial_t\psi^v\rangle d\tau+\alpha_2\int_s^t e^{-\mu(t-\tau)}\langle\zeta,\psi^v\rangle d\tau \\
&\quad+(\beta_1-\beta_2)\int_s^t e^{-\mu(t-\tau)}|\psi^v(\tau)|^2d\tau+\beta_2\int_s^t e^{-\mu(t-\tau)}\langle\eta,\psi^v\rangle d\tau \\
&\quad+\int_s^t e^{-\mu(t-\tau)}(\langle\mathcal{B}(\mathbf{w}_1,\eta)+\mathcal{B}(\zeta,\mathbf{v}_2),\mathcal{A}^{-1}\psi^v\rangle)d\tau=0.
\end{aligned} \tag{29}$$

Doing the difference of (28) and (29) and similar calculations, one proves that:

$$\begin{aligned}\rho^w_\eta(t) - \rho^v_\zeta(t) - & e^{-\mu(t-s)}(\rho^w_\eta(s) - \rho^v_\zeta(s)) \\ & +(\beta_1-\beta_2)\int_s^t e^{-\mu(t-\tau)}(|\psi^w(\tau)|^2 - |\psi^v(\tau)|^2)d\tau \\ & -\int_s^t e^{-\mu(t-\tau)}(\langle P_N\eta, \mathcal{A}^{-1}\partial_t\psi^w\rangle - \langle P_N\zeta, \mathcal{A}^{-1}\partial_t\psi^v\rangle)d\tau \\ & +\alpha_2\int_s^t e^{-\mu(t-\tau)}(\langle\eta,\psi^w\rangle - \langle\zeta,\psi^v\rangle)d\tau \\ & +\beta_2\int_s^t e^{-\mu(t-\tau)}(\langle\zeta,\psi^w\rangle - \langle\eta,\psi^v\rangle)d\tau \\ & +\int_s^t e^{-\mu(t-\tau)}(\langle\mathcal{B}(\mathbf{v}_1,\zeta)+\mathcal{B}(\eta,\mathbf{w}_2), \mathcal{A}^{-1}\psi^w\rangle)d\tau \\ & -\int_s^t e^{-\mu(t-\tau)}(\langle\mathcal{B}(\mathbf{w}_1,\eta)+\mathcal{B}(\zeta,\mathbf{v}_2), \mathcal{A}^{-1}\psi^v\rangle)d\tau = 0.\end{aligned} \tag{30}$$

Thus, similar to (27), we have:

$$\begin{aligned}|\beta_1-\beta_2| & \inf_{[s,t]}\left||\psi^v|^2 - |\psi^w|^2\right| \\ \le & \widetilde{M}\left(\|P_N\eta\|_{C_b(H)} + \|P_N\zeta\|_{C_b(H)}\right) \\ & + \frac{4\sqrt{2}c_P^2\sqrt{\nu'_{\max}}}{\sqrt{\nu_{\min}}N}c_L M_H(\phi^v,\phi^w)M_V(\phi^v,\phi^w)^2(|\alpha_1-\alpha_2|+|\beta_1-\beta_2|),\end{aligned} \tag{31}$$

Summing (27) and (31) then proves (19).

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References

[1] Atte Aalto. Iterative observer-based state and parameter estimation for linear systems. *ESAIM: Control, Optimisation and Calculus of Variations*, 24(1):265–288, 2018.

[2] M. Umer Altaf, Edriss S. Titi, Tarek Gebrael, Omar Knio, L. Zhao, Matthew F. McCabe, and Ibrahim Hoteit. Downscaling the 2D Bénard convection equations using continuous data assimilation. *Computational Geosciences*, 21:393–410, 2017.

[3] Abderrahim Azouani, Eric Olson, and Edriss S Titi. Continuous data assimilation using general interpolant observables. *Journal of Nonlinear Science*, 24:277–304, 2014.

[4] Hakima Bessaih, Eric Olson, and Edriss S Titi. Continuous data assimilation with stochastically noisy data. *Nonlinearity*, 28(3):729, 2015.

[5] Animikh Biswas and Joshua Hudson. Determining the viscosity of the navier–stokes equations from observations of finitely many modes. *Inverse Problems*, 39(12):125012, 2023.

[6] Animikh Biswas, Joshua Hudson, Adam Larios, and Yuan Pei. Continuous data assimilation for the 2D magnetohydrodynamic equations using one component of the velocity and magnetic fields. *Asymptotic Analysis*, 108(1-2):1–43, 2018.

[7] Dirk Blömker, Kody Law, Andrew M Stuart, and Konstantinos C Zygalakis. Accuracy and stability of the continuous-time 3DVAR filter for the Navier–Stokes equation. *Nonlinearity*, 26(8):2193, 2013.

[8] Franck Boyer and Pierre Fabrie. *Mathematical Tools for the Study of the Incompressible Navier–Stokes Equations and Related Models*, volume 183. Springer Science & Business Media, 2012.

[9] Keaton J. Burns, Geoffrey M. Vasil, Jeffrey S. Oishi, Daniel Lecoanet, and Benjamin P. Brown. Dedalus: A flexible framework for numerical simulations with spectral methods. *Physical Review Research*, 2(2):023068, April 2020.

[10] Elizabeth Carlson, Joshua Hudson, and Adam Larios. Parameter recovery for the 2 dimensional Navier–Stokes equations via continuous data assimilation. *SIAM Journal on Scientific Computing*, 42(1):A250–A270, 2020.

[11] Elizabeth Carlson, Joshua Hudson, Adam Larios, Vincent R Martinez, Eunice Ng, and Jared P Whitehead. Dynamically learning the parameters of a chaotic system using partial observations. *Discrete and Continuous Dynamical Systems*, 42(8):3809–3839, 2022.

[12] Emine Celik, Eric Olson, and Edriss S Titi. Spectral filtering of interpolant observables for a discrete-in-time downscaling data assimilation algorithm. *SIAM Journal on Applied Dynamical Systems*, 18(2):1118–1142, 2019.

[13] Walter M Elsasser. The hydromagnetic equations. *Physical Review*, 79(1):183, 1950.

[14] Aseel Farhat, Michael S Jolly, and Edriss S Titi. Continuous data assimilation for the 2D Bénard convection through velocity measurements alone. *Physica D: Nonlinear Phenomena*, 303:59–66, 2015.

[15] Ciprian Foias, Michael S Jolly, Rostyslav Kravchenko, and Edriss S Titi. A determining form for the two-dimensional Navier-Stokes equations: the Fourier modes case. *Journal of Mathematical Physics*, 53(11), 2012.

[16] Ciprian Foias, Michael S Jolly, Rostyslav Kravchenko, and Edriss S Titi. A unified approach to determining forms for the 2D Navier–Stokes equations—the general interpolants case. *Russian Mathematical Surveys*, 69(2):359, 2014.

[17] Ciprian Foias, Cecilia F Mondaini, and Edriss S Titi. A discrete data assimilation scheme for the solutions of the two-dimensional Navier–Stokes equations and their statistics. *SIAM Journal on Applied Dynamical Systems*, 15(4):2109–2142, 2016.

[18] Ciprian Foias and Edriss S Titi. Determining nodes, finite difference schemes and inertial manifolds. *Nonlinearity*, 4(1):135, 1991.

[19] Joshua Hudson and Michael Jolly. Numerical efficacy study of data assimilation for the 2D magnetohydrodynamic equations. *Journal of computational dynamics*, 6(1), 2019.

[20] Hussain A Ibdah, Cecilia F Mondaini, and Edriss S Titi. Fully discrete numerical schemes of a data assimilation algorithm: uniform-in-time error estimates. *IMA Journal of Numerical Analysis*, 40(4):2584–2625, 2020.

[21] Eugenia Kalnay. *Atmospheric modeling, data assimilation and predictability.* Cambridge university press, 2003.

[22] Miroslav Krstic and Andrey Smyshlyaev. *Boundary Control of PDEs.* Society for Industrial and Applied Mathematics, Philadelphia, PA, 2008.

[23] Kody Law, Andrew Stuart, and Kostas Zygalakis. Data assimilation. *Cham, Springer*, 214:52, 2015.

[24] Evelyn Lunasin and Edriss S. Titi. Finite determining parameters feedback control for distributed nonlinear dissipative systems -a computational study. *Evolution Equations and Control Theory*, 6(4):535–557, 2017.

[25] Vincent R Martinez. Convergence analysis of a viscosity parameter recovery algorithm for the 2D Navier–Stokes equations. *Nonlinearity*, 35(5):2241, 2022.

[26] Cecilia F Mondaini and Edriss S Titi. Uniform-in-time error estimates for the postprocessing Galerkin method applied to a data assimilation algorithm. *SIAM Journal on Numerical Analysis*, 56(1):78–110, 2018.

[27] Yuan Pei. Continuous data assimilation for the 3D primitive equations of the ocean. *Communications on Pure and Applied Analysis*, 18(2):643–661, 2019.

[28] James C Robinson. *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, volume 28. Cambridge University Press, 2001.

[29] Michel Sermange and Roger Temam. Some mathematical questions related to the MHD equations. *Communications on Pure and Applied Mathematics*, 36(5):635–664, 1983.

[30] Eduardo D Sontag. *Mathematical control theory: deterministic finite dimensional systems*, volume 6. Springer Science & Business Media, 2013.

[31] Roger Temam. *Navier–Stokes equations and nonlinear functional analysis.* SIAM, 1995.

[32] Roger Temam. *Navier-Stokes equations: theory and numerical analysis*, volume 343. American Mathematical Soc., 2001.

[33] Edriss Saleh Titi. On a criterion for locating stable stationary solutions to the Navier-Stokes equations. *Nonlinear Analysis: Theory, Methods & Applications*, 11(9):1085–1102, 1987.

[34] Hartmut Zohm. *Magnetohydrodynamic stability of tokamaks.* John Wiley & Sons, 2015.